An Numerical Approach to Robust Partial Quadratic Eigenvalue Assignment in Second-Order Control Systems

Zheng-Jian Bai

Department of Information and Computational Mathematics
Xiamen University

*In collaboration with Biswa Datta (Northern Illinois University)

In memory of Professor Gene Golub

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Motivation

Active Vibration Control
  Applications
  Main Goals

Quadratic Eigenvalue Assignment Problem
  Modelling
  Partial Quadratic Eigenvalue Assignment Problem–PQEAP

Minimization Norm and Robust PQEAP
  Difficulties

Existing Methods
  Feedback Norm Minimization Problem
  Robust PQEAP

Our Work
  Parametric Solutions to PQEAP
  Robust PQEAP
  Numerical Results
Figure: Wobbling of the Millennium Footbridge Over the River Thames
Instability of Vibration Structures

- Resonance: The structure is excited by external forces whose frequencies are close to its natural frequencies. The vibrations are amplified and the system becomes unstable.

- Vibration control of structures (e.g. bridges, highways, and aircrafts) are essential to achieve optimal design with desirable performance.

- Traditional Method: Passive damping treatment. Reliable, robust, and without significantly altering the structural mass and stiffness but unadjustable.

Applications of Active Vibration Control

- Large flexible space structure control;
- Earthquake engineering;
- Control of flexible multibody systems;
- Controller design for damped gyroscopic systems;
- Vibration in structural dynamics.
Main Goals of Active Vibration Control

- Determine a state feedback controller to reduce vibrations (e.g. eliminate the resonant frequencies);
- Keep crucial system properties unchanged.
Modelling:

\[ M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = Bu(t). \]

- The dynamical characteristics of a structure are governed by the natural frequencies and mode shapes, i.e., the eigenvalues and eigenvectors of the quadratic eigenvalue problem:

\[ P(\lambda)x := (\lambda^2 M + \lambda C + K)x = 0. \]

- The control force

\[ u(t) = F_1^T \dot{x} + F_2^T x(t) \]

leads to the close-loop pencil

\[ P_c(\lambda) := \lambda^2 M + \lambda(D - BF_1^T) + (K - BF_2^T) = 0. \]
Partial Quadratic Eigenvalue Assignment Problem:

- Find the feedback matrices $F_1$ and $F_2$ such that the few unwanted eigenvalues

$$\lambda_1, \ldots, \lambda_p \ (p \ll 2n)$$

are reassigned to the desired ones:

$$\mu_1, \ldots, \mu_p.$$

- The other $2n - p$ eigenvalues and eigenvectors of the open-loop pencil $P(\lambda)$ are preserved (*no spill-over phenomenon*).
Remarks

- Retaining the “acceptable” (e.g. no-resonant) eigenvalues and eigenvectors can ensure that there is no spurious modes will be introduced into the frequency range of interests;

- One may transform the second-order model to the first-order control system

\[
\dot{z}(t) = \hat{A}z(t) + \hat{B}u(t),
\]

where

\[
\hat{A} = \begin{bmatrix}
0 & I \\
-M^{-1}K & -M^{-1}D
\end{bmatrix}, \quad \hat{B} = \begin{bmatrix}
0 \\
M^{-1}B
\end{bmatrix}, \quad z(t) = \begin{bmatrix}
x(t) \\
\dot{x}(t)
\end{bmatrix}.
\]
Remarks

- The PQEAP is converted the pole assignment problem for the first-order control systems where many numerical methods are available, see Chu’07;

- Computational Drawbacks:
  - Increase in dimensionality from $n$ to $2n$;
  - Inversion of a possibly ill-conditioned mass $M$;
  - Loss of exploitable structures such as the symmetry, definiteness, sparsity and bandedness.

Therefore, the PQEAP is substantially different from the first-order pole assignment problem.
Minimization Norm and Robust PQEAP

From a practical point of view, it is desirable to determine feedback matrices $F_1$ and $F_2$ such that.

- The feedback norm is minimized. Small feedback gains lead to smaller control signals, and thus to less energy consumption. Also, small gains reduce noise amplification.
- The sensitivity of the close-loop eigenvalues is minimized. This is robust PQEAP.
Challenges:

- The problem must be considered in the quadratic setting without the transformation to the first order state space forms.
- Only the few reassigned open-loop eigenvalues and the associated eigenvectors are available.
- No spill-over phenomenon should be guaranteed.
- No model reduction is allowed.
- The gradient formulas of the objective function should be in terms of the few known eigenvalues and eigenvectors only.
Notations

- $\Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_p)$: The eigenvalues to be reassigned;
- $X_1 = [x_1, \ldots, x_p]$: The matrix of the eigenvectors corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_p$;
- $\Lambda'_1 = \text{diag}(\mu_1, \ldots, \mu_p)$: The eigenvalues to assign;
- $\Lambda_2 = \text{diag}(\lambda_{p+1}, \ldots, \lambda_{2n})$: The $2n - p$ open-loop eigenvalues (unavailable);
- $X_2 = [x_{p+1}, \ldots, x_{2n}]$: The matrix of the eigenvectors corresponding to the eigenvalues $\lambda_{p+1}, \ldots, \lambda_{2n}$ (unavailable);
- $Y_1$: Close-loop eigenvectors corresponding to the eigenvalues $\mu_1, \ldots, \mu_p$. 
Assumptions

- $M, D, K$ real symmetric with $M > 0$.
- Partial Controllability of the model $(P(\lambda), B)$
  \[ \text{rank}(\lambda_i^2 M + \lambda_i D + K, B) = n \text{ for } i = 1, \ldots, p. \]
- Sets of eigenvalues are closed under complex conjugation.
- $0 \not\in \{\lambda_1, \ldots, \lambda_p\}$. 
Parametric Expressions for the feedback Matrices

(Theorem: Brahma and Datta’07) The feedback matrices $F_1$ and $F_2$ are determined by

\[
\begin{align*}
F_1 &= MX_1 \Lambda_1 \Phi^T \\
F_2 &= -KX_1 \Phi^T
\end{align*}
\]

where $\Phi$ satisfies

$$\Phi Z = \Gamma \text{ (\Gamma arbitrary)}$$

with $Z$ being the unique solution of the Sylvester equation:

$$\Lambda_1 Z - Z \Lambda_1' = -\Lambda_1 X_1^T B \Gamma.$$
Feedback Norm Minimization Problem

Let

\[ S := [F_2^T, F_1^T] \]

\[
\min \ \Pi := \frac{1}{2} \| S \|_F^2 = \frac{1}{2} (\| F_1 \|_F^2 + \| F_2 \|_F^2)
\]

- Theorem for Gradient Formula (Brahma and Datta’07):
  - Let \( C := [-X_1^T K, \Lambda_1 X_1^T M] \). Then
    \[ S = \Gamma Z^{-1} C \]
  - Let \( U \) be the solution to the Sylvester equation
    \[ \Lambda'_1 U - U \Lambda_1 = -Z^{-1} CS^H \Phi. \]
    Then,
    \[ \nabla_\Gamma (\Pi) = 1/2[Z^{-1} CS^H - U \Lambda_1 X_1^T B]^T. \]
Robust PQEAP

The close-loop eigenvectors

\[ Y = \begin{bmatrix} Y_1 & X_2 \\ Y_1\Lambda'_1 & X_2\Lambda_2 \end{bmatrix} \]

where \( Y_1 = [y_1, \ldots, y_p] \) with \( y_k \) satisfying \((\mu_k^2 M + \mu_k D + K)y_k = B\gamma_k\).

\[
\min \quad J := \| (I - Y^H Y)^2 \|_F^2
\]

- **Theorem for Gradient Formula (Brahma and Datta’07):**
  - \( J = \| (I - Y^H Y)^2 \|_F^2 = \| Z_1 \|_F^2 + \| Z_2 \|_F^2 := J_1 + J_2 \), where
    \[
    Z_1 = I_p - Y_1^H Y_1 - \Lambda'_1 Y_1^H Y_1 \Lambda_1, \quad Z_2 = I_{2n-p} - X_2^H X_2 - \Lambda_2 X_2^H X_2 \Lambda_2.
    \]
  - \[
  \nabla_{\Gamma}(J) = \nabla_{\Gamma}(J_1) \quad (\nabla_{\Gamma}(J_2) = 0),
  \]
  where \( \nabla_{\Gamma}(J_1) \) is given in terms of \( Y_1, X_1, \) and \( \Lambda_1 \) only.
Attn: The assumption that \( 0 \not\in \{\lambda_1, \ldots, \lambda_p\} \) is canceled.

- The feedback matrices \( F_1 \) and \( F_2 \) are given by

\[
\begin{align*}
F_1 &= MX_1 \Phi^T \\
F_2 &= (MX_1 \Lambda_1 + DX_1) \Phi^T
\end{align*}
\]

where \( \Phi \) satisfies

\[
\Phi Z = \Gamma \quad (\Gamma \text{ arbitrary})
\]

with \( Z \) being the unique solution of the Sylvester equation:

\[
\Lambda_1 Z - Z \Lambda_1' = -X_1^T B \Gamma.
\]
Robust PQEAP

The close-loop eigenvectors

\[ Y = \begin{bmatrix} Y_1 & X_2 \\ Y_1 \Lambda_1' & X_2 \Lambda_2 \end{bmatrix} \]

where \( Y_1 = [y_1, \ldots, y_p] \) with \( y_k \) satisfying \((\mu_k^2 M + \mu_k D + K)y_k = B \gamma_k\).

\[
\min J := \frac{1}{2}(1 - \alpha)(\|F_1\|_F^2 + \|F_2\|_F^2) + \frac{1}{2}\alpha(\|Y\|_F^2 + \|Y^{-1}\|_F^2),
\]

where \( 0 \leq \alpha \leq 1 \).
Remarks

- Reduction of $J$:

\[
J = \frac{1}{2} \alpha (\| Y_1 \|^2_F + \| Y_1 \Lambda_1' \|^2_F) + \frac{1}{2} \alpha \| Y^{-1} \|^2_F \\
+ \frac{1}{2} (1 - \alpha) (\| F_1 \|^2_F + \| F_2 \|^2_F) \\
+ \frac{1}{2} \alpha (\| X_2 \|^2_F + \| X_2 \Lambda_2 \|^2_F) \\
:= J_1 + J_2 + J_3.
\]

- Challenges:
  - Could we get an explicit expression for $Y^{-1}$?
  - How to find the gradient formula for the above cost function without knowing $\Lambda_2$ and $X_2$?

- If the formula is found, then an optimization technique such as Broyden-Fletcher-Goldfrab-Shannon (BFGS) method can be employed.
Expression of $Y^{-1}$

**Theorem:** Let

\[
\begin{align*}
C & := [\Lambda_1 X_1^T M + X_1^T D, X_1^T M], \\
\tilde{Y}_1 & := [Y_1^T, \Lambda_1' Y_1^T]^T, \\
\tilde{X}_2 & := [X_2^T, \Lambda_2 X_2^T]^T.
\end{align*}
\]

Then

\[
Y^{-1} = \begin{bmatrix}
Z^{-1} C \\
\tilde{X}_2^+ (I - \tilde{Y}_1 Z^{-1} C)
\end{bmatrix} := \begin{bmatrix}
W_1 \\
W_2
\end{bmatrix}.
\]
Comments

It follows that

\[ \| W_2 \|_F \leq \| \tilde{X}_2^+ \|_2 \| (I - \tilde{Y}_1 W_1) \|_F = \frac{1}{\sigma_{\min}(\tilde{X}_2)} \| I - \tilde{Y}_1 W_1 \|_F. \]

By the assumption that \( \tilde{X}_2 \) has linearly independent columns, one may expect that \( \sigma_{\min}(\tilde{X}_2) \) is not too small. Suppose that \( \beta \) is \textit{a-priori} estimate of \( 1/\sigma_{\min}^2(\tilde{X}_2) \). Therefore, we shall minimize

\[
J = \frac{1}{2} \alpha \left( \| Y_1 \|_F^2 + \| Y_1 \Lambda_1' \|_F^2 + \| W_1 \|_F^2 + \beta \| I - \tilde{Y}_1 W_1 \|_F^2 \right) \\
+ \frac{1}{2} (1 - \alpha) \left( \| F_1 \|_F^2 + \| F_2 \|_F^2 \right) \\
+ \frac{1}{2} \alpha \left( \| X_2 \|_F^2 + \| X_2 \Lambda_2 \|_F^2 \right) := J_1 + J_2 + J_3 \quad \text{(for simplicity)}.
\]
Gradient Formula

- **Theorem:** Let $U$, $U_1$, $U_2$, $\gamma$, $V$, $V_1$, and $V_2$, respectively, satisfy the following equations

$$
\begin{align*}
MU\Lambda_1^{'2} + DU\Lambda_1' + KU &= [(I + \Lambda_1' \bar{\Lambda}_1') Y_1^H]^T, \\
MU_1\Lambda_1^{'2} + DU_1\Lambda_1' + KU_1 &= [W_1 W_{11}^H]^T, \\
MU_2\Lambda_1^{'2} + DU_2\Lambda_1' + KU_2 &= [\Lambda_1' W_1 W_{21}^H]^T
\end{align*}
$$

and

$$
\begin{align*}
\Lambda_1' \gamma - \gamma \Lambda_1 &= -Z^{-1} CS^H \Phi \\
\Lambda_1' V - V \Lambda_1 &= -W_1 W_{11}^H Z^{-1}, \\
\Lambda_1' V_1 - V_1 \Lambda_1 &= -W_1 W_{11}^H Y_1 Z^{-1}, \\
\Lambda_1' V_2 - V_2 \Lambda_1 &= -W_1 W_{21}^H Y_1 \Lambda_1' Z^{-1},
\end{align*}
$$
Gradient Formula

where

\[ W_{11} = E_1 - Y_1 W_1 \quad \text{and} \quad W_{21} = E_2 - Y_1 \Lambda_1^T W_1 \]

with \( E_1 := [I_n, 0] \in \mathbb{R}^{n \times 2n} \) and \( E_2 := [0, I_n] \in \mathbb{R}^{n \times 2n} \). Then,

\[ \nabla_{\Gamma} J = \alpha \nabla_{\Gamma} J_1 + (1 - \alpha) \nabla_{\Gamma} J_2, \]

where

\[ \nabla_{\Gamma} J_1 = \frac{1}{2} [U^T B + VX_1^T B]^T \]

\[ - \frac{1}{2} \beta [(U_1 + U_2)^T B + (V_1 + V_2)X_1^T B]^T, \]

\[ \nabla_{\Gamma} J_2 = \frac{1}{2} [Z^{-1} CS^H - \Upsilon X_1^T B]^T. \]
Numerical Results

Problem 1:

- The open-loop eigenvalues are:
  \{-3.4209, -0.1943 \pm 1.0642i, 0.0000, -0.5474 \pm 0.8820i, -1.0777, 0.0000\}. The eigenvalues \{0.0000, 0.0000\} were reassigned to \{-0.3 \pm 1.5i\}, the other eigenvalues were retained.
Numerical Results

Problem 2 (Nichols & Kautsky’01):

\[ M = 10I_3, \quad D = 0, \quad K = \begin{bmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{bmatrix}. \]

- The open-loop eigenvalues are: \( \{ \pm 3.6039i, \pm 2.4940i, \pm 0.8901i \} \).
- The first two eigenvalues \( \{ \pm 3.6039i \} \) were reassigned to \( \{-1, -2\} \), the other eigenvalues were preserved.
Numerical Results

Problem 3 (Tisseur & Meerbergen’01):

- \( M = I_n \), \( D = \tau \text{tridiag}(-1, 3, -1) \), \( K = \kappa \text{tridiag}(-1, 3, -1) \), \( B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -1 \end{bmatrix} \)

- \( n = 50 \), \( \tau = 3 \), and \( \kappa = 5 \).
- The first six open-loop complex eigenvalues \( \{-3.3306 \pm 0.0947i, -3.1628 \pm 0.7344i, -3.0000 \pm 1.0000i\} \) were reassigned to \( \{-3.5 \pm 0.0947i, -3.5 \pm 0.7344i, -3.5 \pm 1.0000i\} \). The other 94 eigenpairs are kept unchanged.
**Table:** Numerical results for Problems 1–3 ("e5" means “×10^5”)

<table>
<thead>
<tr>
<th></th>
<th>Mini-N</th>
<th>( \beta = 1 )</th>
<th>( \beta = 10^2 )</th>
<th>( \beta = 10^4 )</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>( \alpha = 0 )</td>
<td>( \alpha = 1 )</td>
<td>( \alpha = 0.5 )</td>
<td>( \alpha = 1 )</td>
</tr>
<tr>
<td>( | F_1 |_F )</td>
<td>20.09</td>
<td>12.88</td>
<td>12.38</td>
<td>21.78</td>
</tr>
<tr>
<td>( | F_2 |_F )</td>
<td>23.40</td>
<td>12.19</td>
<td>12.73</td>
<td>51.05</td>
</tr>
<tr>
<td>( \kappa_2(Y) )</td>
<td>6.8e12</td>
<td>22.03</td>
<td>24.17</td>
<td>162.81</td>
</tr>
<tr>
<td></td>
<td>( | F_1 |_F )</td>
<td>19.36</td>
<td>26.58</td>
<td>21.31</td>
</tr>
<tr>
<td></td>
<td>( | F_2 |_F )</td>
<td>70.89</td>
<td>99.68</td>
<td>72.23</td>
</tr>
<tr>
<td></td>
<td>( \kappa_2(Y) )</td>
<td>4.86e3</td>
<td>10.19</td>
<td>177.05</td>
</tr>
<tr>
<td></td>
<td>( | F_1 |_F )</td>
<td>5.24</td>
<td>18.23</td>
<td>23.41</td>
</tr>
<tr>
<td></td>
<td>( | F_2 |_F )</td>
<td>18.22</td>
<td>74.04</td>
<td>92.41</td>
</tr>
<tr>
<td></td>
<td>( \kappa_2(Y) )</td>
<td>2.2e9</td>
<td>2.9e5</td>
<td>4.2e5</td>
</tr>
</tbody>
</table>
Table: Numerical results for Problems 2 and 3 ($\text{RE.} = 100 \times (\text{IV.} - \text{FV.})/\text{IV.}$)

<table>
<thead>
<tr>
<th></th>
<th>Brahma-Datta’s Robust Method</th>
<th>Our method with $\alpha = 1$, $\beta = 10^2$</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$|F_1|_F$</td>
<td>$|F_2|_F$</td>
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<tr>
<td>IV.</td>
<td>74.98</td>
<td>130.71</td>
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<tr>
<td>FV.</td>
<td>59.44</td>
<td>161.54</td>
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<tr>
<td>RE.</td>
<td>20.72</td>
<td>$-23.58$</td>
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<tr>
<td>IV.</td>
<td>5.1067</td>
<td>17.83</td>
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<td>$-22.46$</td>
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Thank You!