Symmetric Tridiagonal Inverse Quadratic Eigenvalue Problems
With Partial Eigendata

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Introduction

In vibration and structural analysis, we often need to solve a linear second-order differential equation (e.g. Finite Element Model)

\[ M{\ddot{x}}(t) + C{\dot{x}}(t) + Kx(t) = 0, \]

where \( M, C, K \in \mathbb{C}^{n \times n} \) and \( x(t) \) is an \( n \)th-order vector. The separation of variables \( x(t) = xe^{\lambda t} \) gives rise to the quadratic eigenvalue problem (See Tisseur'01)

\[ Q(\lambda)x \equiv (\lambda^2 M + \lambda C + K)x = 0. \]
Inverse Quadratic eigenvalue Problems (IQEP):

- **Predicted** frequencies & model shapes(eigenvalues/eigenvectors) often disagree with that of experimentally measured from a realizable practical structure

- Reconstructing the quadratic pencil

\[ Q(\lambda) \equiv \lambda^2 M + \lambda C + K \]

from experimentally measured frequencies/mode shapes.
Tridiagonal IQEPs

- Construct a nontrivial quadratic pencil

\[ Q(\lambda) = \lambda^2 I + \lambda C + K \]

from a set of measured eigendata \( \{ (\lambda_i, x_i) \}_{i=1}^{p} \), where

\[
C = \begin{bmatrix}
a_1 & -b_2 & & & \\
-\ b_2 & a_2 & -b_3 & & \\
& \ddots & \ddots & \ddots & \\
& & -b_n & a_n & \\
\end{bmatrix}, \quad
K = \begin{bmatrix}
c_1 & -d_2 & & & \\
-\ d_2 & c_2 & -d_3 & & \\
& \ddots & \ddots & \ddots & \\
& & -d_n & c_n & \\
\end{bmatrix}
\]
Applications: Vibration Systems (Nylen99, Gladwell04)

Difficulty: In practice, the physically realizable parameters \( \{a_i\}_{1}^{n} \), \( \{b_i\}_{2}^{n} \), \( \{c_i\}_{1}^{n} \), and \( \{d_i\}_{2}^{n} \) should be \textbf{positive} such that the corresponding \( C \) and \( K \) should be \textbf{weakly diagonally dominant}.

- Ram and Elhay (1996) determined the parameters from two sets of eigenvalues, where the positiveness is not necessarily preserved.
Problem Reformulation

Given the eigendata \((\Lambda, X) \in \mathbb{R}^{p \times p} \times \mathbb{R}^{n \times p}\) with

\[
\Lambda = \text{diag}\{\lambda_1^{[2]}, \ldots, \lambda_s^{[2]}, \lambda_{s+1}, \ldots, \lambda_p\}, \quad \lambda_i^{[2]} = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \in \mathbb{R}^{2 \times 2}, \quad 1 \leq i \leq s,
\]

\[
X = [x_{1R}, x_{1I}, \ldots, x_{sR}, x_{sI}, x_{s+1}, \ldots, x_p]
\]

The IQEP becomes

\[
\begin{align*}
\min & \quad \frac{1}{2}\|C - C_a\|^2 + \frac{1}{2}\|K - K_a\|^2 \\
\text{s.t.} & \quad X\Lambda^2 + CX\Lambda + KX = 0, \\
& \quad C, K \in \Omega.
\end{align*}
\]
\[ \Omega: \text{The set of tridiagonal and weakly diagonally dominant matrices with the positive diagonals and negative off-diagonals.} \]

Attn: The eigendata \((\Lambda, X)\) are experimentally measured which often corrupted by noise.

To reduce the sensitivity, we solve the quadratically constrained quadratic problem (QCQP)

\[
\begin{align*}
\min & \quad \frac{1}{2} \| C - C_a \|^2 + \frac{1}{2} \| K - K_a \|^2 \\
\text{s.t.} & \quad \| X \Lambda^2 + CX \Lambda + KX \| \leq \delta_n, \\
& \quad C, K \in \Omega.
\end{align*}
\]

where \(\delta_n\) is a positive parameter depending on the noise level of the measured eigendata. Attn: \(\delta_n \to 0\), problem (1) is recovered.
Since \( C, K \in \Omega \), Problem (2) is reduced to

\[
\min \quad f_0(y) := \frac{1}{2} \| y - y^o \|^2 \\
s.t. \quad f_1(y) := \| Ay - g \|^2 - \delta^2_n \leq 0, \\
\quad f_2(y) := By \geq 0, \\
\quad y \geq 0,
\]

where the matrix \( A \) and the vector \( g \) are given in terms of the eigendata, \( By \geq 0 \) corresponds to the weakly diagonally dominant constraint, where

\[
y_o = (y^1_o, y^2_o, \ldots, y^n_o)^T \in \mathbb{R}^{4n-2} \text{ with } y^1_o = (a^o_1, c^o_1)^T \text{ and } y^i_o = (a^o_i, b^o_i, c^o_i, d^o_i)^T \text{ for } 2 \leq i \leq n, \\
y = (y^1, y^2, \ldots, y^n)^T \in \mathbb{R}^{4n-2} \text{ with } y^1 = (a_1, c_1)^T \text{ and } y^i = (a_i, b_i, c_i, d_i)^T \text{ for } 2 \leq i \leq n.
\]
• Problem (3) is equivalent to finding $y \geq 0$, $\xi \geq 0$, and $\zeta \geq 0$ such that

$$
\nabla f_0(y) + \xi \nabla f_1(y) - \nabla f_2(y)^T \zeta = 0,
$$

\[ \text{KKT: } \begin{cases} 
\xi \geq 0, & \zeta \geq 0, & -f_1(y) \geq 0, & f_2(y) \geq 0, \\
\xi f_1(y) = 0, & f_2(y)^T \zeta = 0.
\end{cases} \tag{4}
\]

Let

$$
F(z) := \begin{pmatrix} 
\nabla f_0(y) + \xi \nabla f(y) - \nabla f_2(y)^T \zeta \\
-f_1(y) \\
f_2(y)
\end{pmatrix}, \quad z := (y, \xi, \zeta)
$$

Solving (4) is to find a vector $z^* \in \mathcal{K}$ such that
Variational inequalities: \((z - z^*)^T F(z^*) \geq 0\), for all \(z \in \mathcal{K}\) \hspace{1cm} (5)

or Robinson’s normal equation:

\[
F_0(z) := F(\Pi_{\mathcal{K}}(z)) + z - \Pi_{\mathcal{K}}(z) = 0.
\] \hspace{1cm} (6)

in the sense that if \(\hat{z}^*\) is a solution of (6), then \(\underline{z^*} := \Pi_{\mathcal{K}}(\hat{z}^*)\) is a solution of (5), and conversely if \(z^*\) is a solution of (5), then \(\hat{z}^* := z^* - F(z^*)\) is a solution of (6), see (Robinson92).

**Smoothing Approx**: By using the Chen-Harker-Kanzow-Smale smoothing function (Chen & Harker93) for \(\Pi_{\mathcal{K}}(\cdot)\), we get the smoothing approximation for \(F_0(\cdot)\):
\[ \tilde{G}(\epsilon, z) := F(p(\epsilon, z)) + z - p(\epsilon, z), \quad (\epsilon, z) \in \mathbb{R} \times \mathbb{R}^m \]  

(7)

where

\[ p(\epsilon, z) = \text{vec}\{\varphi(\epsilon, z_i)\}, \]
\[ \varphi(\epsilon, x) := \frac{1}{2} \left( x + \sqrt{x^2 + 4\epsilon^2} \right) \rightarrow x_+ \text{ as } \epsilon \rightarrow 0, \quad (\epsilon, x) \in \mathbb{R} \times \mathbb{R}. \]

To prevent the singularity, we define the regularized function

\[ H : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{m+1} \] by
$H(w) := \begin{pmatrix} \epsilon \\ G(w) \end{pmatrix}, \quad w := (\epsilon, z) \in \mathbb{R} \times \mathbb{R}^m,$

\[ G(w) := \tilde{G}(w) + \epsilon z. \]  

$z^* = (y^*, \xi^*, \zeta^*)$ solves (6) $\iff w^* = (0, z^*)$ solves $H(w) = 0$
Regularized Smoothing Newton Method

- $\bar{\epsilon} \in \mathbb{R}_{++}$ and $\tau \in (0, 1)$ such that $\tau \bar{\epsilon} < 1$.

- $\bar{w} := (\bar{\epsilon}, 0, 0) \in \mathbb{R} \times \mathbb{R}^{3n} \times \mathbb{R}$.

- $\phi(w) := \|H(w)\|^2$ and $\psi(w) := \tau \min(1, \phi(w))$.

\[
H(w^{(k)}) + H'(w^{(k)}) \Delta w^{(k)} = \psi(w^{(k)}) \bar{w}
\]

\[
\phi(w^{(k)}) + \delta^l \Delta w^{(k)} \leq [1 - 2\sigma(1 - \tau \bar{\epsilon})\delta^l] \phi(w^{(k)})
\]
Our Work

- **Theorem:** $\forall w = (\epsilon, z) \in \mathbb{R}_+ \times \mathbb{R}^m$, $H'(w)$ is nonsingular.

- **Global Convergence:** Suppose the solution set of (6) is nonempty. Then the infinite sequence $\{w^{(k)}\}$ generated by our algorithm is bounded and any accumulation point $w^*$ of $\{w^{(k)}\}$ is a solution of $H(w) = 0$.

- **Quadratic Convergence:** Suppose that $w^*$ is an accumulation point of the sequence $\{w^{(k)}\}$ generated by our algorithm. If all $V \in \partial H(w^*)$ are nonsingular, then the whole sequence $\{w^{(k)}\}$ converges to $w^*$ with

$$
\|w^{(k+1)} - w^*\| = O(\|w^k - w^*\|^2), \quad \epsilon^{(k+1)} = O((\epsilon^{(k)})^2).
$$
Numerical Results in Engineering Applications

- A damped mass-spring system governed by (see also Ram and Elhay'96)

\[ I\ddot{x}(t) + C\dot{x}(t) + Kx(t) = 0, \]

\[ C = P\text{diag}(0, e_1, e_2, \ldots, e_{n-1}) P^T + \text{diag}(\pi_1, \ldots, \pi_n) \]
\[ K = P\text{diag}(0, f_1, f_2, \ldots, f_{n-1}) P^T + \text{diag}(\kappa_1, \ldots, \kappa_n) \]

where \( P = [\delta_{ij} - \delta_{i+1,j}] \).

- \( \{e_i^o\}_{1}^{n-1} \), \( \{\pi_i^o\}_{1}^{n} \), \( \{f_i^o\}_{1}^{n-1} \), \( \{\kappa_i^o\}_{1}^{n} \), and \( \{\lambda_i, x_i\}_{1}^{p} \) generated randomly.
• An upper bound estimate for the noise parameter $\delta_n$ (See Abdalla, Grigoriadis, and Zimmerman'00):

$$\delta_n = r(\|X\Lambda_2\| + \|C_0X\Lambda\| + \|K_0X\|), \quad r = 0.08$$

$a) : \epsilon^{(0)} = \bar{\epsilon}, \quad z^{(0)} = 0; \quad b) : \epsilon^{(0)} = \bar{\epsilon}, \quad z^{(0)} = 1, \quad \bar{\epsilon} = 0.1$.

$\delta = 0.5, \quad \sigma = 0.5 \times 10^{-4}, \quad \tau = 0.2 \times \min(1, 1/\bar{\epsilon})$.  

**Stopping criterion:** $\|H(w^{(k)})\| \leq 10^{-6}$
\( p = 15, s = 3 \)

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Conclusions

- A tridiagonal inverse quadratic eigenvalue problem is discussed.

- The problem is reformulated as a variational inequality.

- A regularized smoothing Newton method is proposed.

- Numerical experiments show the efficiency of our approach.
Thank You!