A Note on Simple Nonzero Finite Generalized Singular Values

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Abstract

In this paper, we study the sensitivity and second order perturbation expansions of simple nonzero finite generalized singular values of a complex matrix pair, which is analytically dependent on several parameters. Our results generalize the perturbation analysis given by Sun [J. Comput. Math., 6 (1988), pp. 258–266] for simple nonzero singular values.

Keywords Generalized singular value, sensitivity analysis

AMS subject classification. 65F15, 15A18

1 Introduction

The generalized singular value decomposition (GSVD) of two matrices of the same number of columns was proposed by Van Loan [25] and further studied by Paige and Saunders [13]. The GSVD is very useful in many applications including constrained least squares problems [8, p. 580] and [25], weighted least squares problems [4, 25], information retrieval [10], linear discriminant analysis [14], computing the Kronecker structure of matrix pencil $A - \lambda B$ [12], discriminant analysis [11], ionospheric tomography [3], etc. Numerical methods and sensitivity analysis of the GSVD can be found, for instance, in [1, 6, 13, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26].

Recently, Chen and Li [7] presented the sensitivity of multiple nonzero finite generalized singular values of a real matrix pair, which is analytically dependent on several parameters.

In this paper, we focus on the sensitivity issue and second order perturbation expansions of simple nonzero finite generalized singular values of a complex matrix pair analytically dependent on several parameters. We provide the explicit expressions for the first order partial derivatives of simple nonzero finite generalized singular values of the complex matrix pair and associated generalized singular matrix set. We also give the second order partial derivatives of simple nonzero finite generalized singular values of the complex matrix pair, which is very useful for computing the second order Taylor expansions for simple nonzero finite generalized singular

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values. Our results may be used to check the effectiveness and stability of GSVD-based methods for practical applications such as the method of particular solutions for solving planar eigenvalue problems [2], ionospheric tomography techniques [3], and discriminant analysis [11], etc. Our results generalize the perturbation analysis in [21] for simple non-zero singular values.

Throughout this paper, the following notation will be used. Let $\mathbb{C}^{m\times n}$ and $\mathbb{R}^{m\times n}$ stand for the set of all $m\times n$ complex matrices and the set of all $m\times n$ real matrices, respectively. The symbols $\mathbb{C}^n$ and $\mathbb{R}^n$ denote the set of all complex n-vectors and the set of all real n-vectors, respectively. Denote by A^T and A^H the transpose and the conjugate transpose of a matrix A, respectively. Let I_n be the identity matrix of order n.

The paper is organized as follows. In Section 2 we introduce some preliminary results on the GSVD. In Section 3 we discuss the sensitivity analysis and second order perturbation expansions of simple nonzero finite generalized singular values. The partial derivatives of a generalized singular vector set corresponding to the simple nonzero finite generalized singular values are also established. In Section 4 we define the sensitivity of simple nonzero finite generalized singular values and give some examples for computing the sensitivity and second order expansions of simple nonzero finite generalized singular values. Finally, some conclusions and future work are presented in Section 5.

2 Preliminaries

On the GSVD of a complex matrix pair, we have the following result [13, 25].

Lemma 2.1 *Let $A\in\mathbb{C}^{m\times n}$ and $B\in\mathbb{C}^{l\times n}$ be such that* $\mathrm{rank}(A^H,B^H)=n$. *Then, there exist two unitary matrices $U\in\mathbb{C}^{m\times m}$, $V\in\mathbb{C}^{l\times l}$ and a nonsingular matrix $Q\in\mathbb{C}^{n\times n}$ such that*

$$U^HAQ=\begin{pmatrix}\Lambda & \\ & 0_{(m-r-s)\times(n-r-s)}\end{pmatrix}\quad \textit{and}\quad V^HBQ=\begin{pmatrix}0_{(l+r-n)\times r} & \\ & \Sigma\end{pmatrix},$$

where $0_{s\times t}$ denotes the $s\times t$ null matrix and

$$\Lambda=\mathrm{diag}(\alpha_1,\ldots,\alpha_{r+s})\quad \textit{and}\quad \Sigma=\mathrm{diag}(\beta_{r+1},\ldots,\beta_n)$$

with

$$\begin{aligned}1=\alpha_1=\cdots=\alpha_r>\alpha_{r+1}\geq\cdots\geq\alpha_{r+s}>\alpha_{r+s+1}=\cdots=\alpha_n=0,\\ 0=\beta_1=\cdots=\beta_r<\beta_{r+1}\leq\cdots\leq\beta_{r+s}<\beta_{r+s+1}=\cdots=\beta_n=1,\\ \alpha_j^2+\beta_j^2=1\quad \textit{for}\quad 1\leq j\leq n.\end{aligned}$$

Here, $\{(\alpha_j,\beta_j)\}_{j=1}^n$ are called the generalized singular values of the complex matrix-pair $\{A,B\}$. For simplicity, denote by $\sigma\{A,B\}$ the set of generalized singular values of $\{A,B\}$.

Suppose that (α,β) is a simple nonzero finite generalized singular value of $\{A,B\}$ and $\sigma_1=\alpha/\beta$. Then, By Lemma 2.1, we can easily know that there exist two unitary matrices $U=(\mathbf{u}_1,U_2)\in\mathbb{C}^{m\times m}, V=(\mathbf{v}_1,V_2)\in\mathbb{C}^{l\times l}$ and a nonsingular matrix $Q=(\mathbf{q}_1,Q_2)\in\mathbb{C}^{n\times n}$ with $\mathbf{u}_1\in\mathbb{C}^m,\mathbf{v}_1\in\mathbb{C}^l$ and $\mathbf{q}_1\in\mathbb{C}^n$ such that

$$U^HAQ=\begin{pmatrix}\sigma_1 & 0\\ 0 & \Lambda_2\end{pmatrix}\quad\text{and}\quad V^HBQ=\begin{pmatrix}1 & 0\\ 0 & \Sigma_2\end{pmatrix},\tag{1}$$

where $(\sigma_1, 1) \notin \sigma\{\Lambda_2, \Sigma_2\}$. Therefore, we get by (1),

$$A\mathbf{q}_1 = \sigma_1 \mathbf{u}_1, \quad B\mathbf{q}_1 = \mathbf{v}_1, \quad \mathbf{u}_1^H \mathbf{u}_1 = \mathbf{v}_1^H \mathbf{v}_1 = 1. \tag{2}$$

The vector set $\{\mathbf{u}_1, \mathbf{v}_1, \mathbf{q}_1\}$ satisfying (2) is called a generalized singular vector set of $\{A, B\}$ corresponding to the generalized singular value $(\sigma_1, 1)$.

3 Computing partial derivatives

In this section, we study the sensitivity issue of simple nonzero finite generalized singular values and the associated generalized singular vector set.

Let $\mathbf{p} = (p_1, \ldots, p_N)^T$, $A(\mathbf{p}) \in \mathbb{C}^{m\times n}$, and $B(\mathbf{p}) \in \mathbb{C}^{l\times n}$. In what follows, without loss of generality, we assume that $\mathbf{p} \in \mathbb{R}^N$.

By using the Implicit Function Theorem, we establish the following results.

Theorem 3.1 *Let $\mathbf{p} \in \mathbb{R}^N$, $A(\mathbf{p}) \in \mathbb{C}^{m\times n}$, $B(\mathbf{p}) \in \mathbb{C}^{l\times n}$, and $\operatorname{rank}(A(\mathbf{p})^H, B(\mathbf{p})^H) = n$. Suppose that $\operatorname{Re}[A(\mathbf{p})]$, $\operatorname{Re}[B(\mathbf{p})]$, $\operatorname{Im}[A(\mathbf{p})]$ and $\operatorname{Im}[A(\mathbf{p})]$ are all real analytic matrix-valued functions of $\mathbf{p}$ in some neighborhood $\mathscr{B}(\mathbf{0})$ of the origin. If $(\sigma_1, 1)$ $(\sigma_1 > 0)$ is a simple nonzero finite generalized singular value of $\{A(\mathbf{0}), B(\mathbf{0})\}$ and there exist two unit vectors $\mathbf{u}_1 \in \mathbb{C}^m, \mathbf{v}_1 \in \mathbb{C}^l$ and a nonzero vector $\mathbf{q}_1 \in \mathbb{C}^n$ such that $\{\mathbf{u}_1, \mathbf{v}_1, \mathbf{q}_1\}$ is an associated generalized singular vector set of $\{A(\mathbf{0}), B(\mathbf{0})\}$, then*

1) *there exists a simple generalized singular value $(\sigma_1(\mathbf{p}), 1)$ of $\{A(\mathbf{p}), B(\mathbf{p})\}$ such that $\sigma_1(\mathbf{p})$ is a real analytic function of $\mathbf{p}$ in some neighborhood $\mathscr{B}_0 \subset \mathscr{B}(\mathbf{0})$ of the origin, $\sigma_1(\mathbf{p}) > 0$, and $\sigma_1(\mathbf{0}) = \sigma_1$.*

2) *there exist $\mathbf{u}_1(\mathbf{p}) \in \mathbb{C}^m, \mathbf{v}_1(\mathbf{p}) \in \mathbb{C}^l$, and $\mathbf{q}_1(\mathbf{p}) \in \mathbb{C}^n$ with $\mathbf{u}_1(\mathbf{p})^H \mathbf{u}_1(\mathbf{p}) = \mathbf{v}_1(\mathbf{p})^H \mathbf{v}_1(\mathbf{p}) = 1$ such that $\operatorname{Re}[\mathbf{u}_1(\mathbf{p})]$, $\operatorname{Im}[\mathbf{u}_1(\mathbf{p})]$, $\operatorname{Re}[\mathbf{v}_1(\mathbf{p})]$, $\operatorname{Im}[\mathbf{v}_1(\mathbf{p})]$, $\operatorname{Re}[\mathbf{q}_1(\mathbf{p})]$, and $\operatorname{Im}[\mathbf{q}_1(\mathbf{p})]$ are all real analytic functions of $\mathbf{p}$ in $\mathscr{B}_0$ and $\{\mathbf{u}_1(\mathbf{p}), \mathbf{v}_1(\mathbf{p}), \mathbf{q}_1(\mathbf{p})\}$ is a generalized singular vector set of $\{A(\mathbf{p}), B(\mathbf{p})\}$ corresponding to the generalized singular value $(\sigma_1(\mathbf{p}), 1)$, i.e.,*

$$A(\mathbf{p})\mathbf{q}_1(\mathbf{p}) = \sigma_1(\mathbf{p})\mathbf{u}_1(\mathbf{p}) \quad \text{and} \quad B(\mathbf{p})\mathbf{q}_1(\mathbf{p}) = \mathbf{v}_1(\mathbf{p}),$$

where $\mathbf{u}_1(\mathbf{0}) = \mathbf{u}_1, \mathbf{v}_1(\mathbf{0}) = \mathbf{v}_1$ and $\mathbf{q}_1(\mathbf{0}) = \mathbf{q}_1$.

Proof: By the hypothesis, there exist two unitary matrices $U = (\mathbf{u}_1, U_2) \in \mathbb{C}^{m\times m}, V = (\mathbf{v}_1, V_2) \in \mathbb{C}^{l\times l}$, and a nonsingular matrix $Q = (\mathbf{q}_1, Q_2) \in \mathbb{C}^{n\times n}$ such that

$$U^H A(\mathbf{0}) Q = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \quad \text{and} \quad V^H B(\mathbf{0}) Q = \begin{pmatrix} 1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}, \tag{3}$$

where $(\sigma_1, 1) \notin \sigma(\Lambda_2, \Sigma_2)$ and $\sigma_1 > 0$.

Let

$$\tilde{A}(\mathbf{p}) = Q^H A(\mathbf{p})^H A(\mathbf{p}) Q = \begin{pmatrix} \tilde{a}_{11}(\mathbf{p}) & \tilde{a}_{21}(\mathbf{p})^H \\ \tilde{a}_{21}(\mathbf{p}) & \tilde{A}_{22}(\mathbf{p}) \end{pmatrix}, \tag{4}$$

$$\tilde{B}(\mathbf{p}) = Q^H B(\mathbf{p})^H B(\mathbf{p}) Q = \begin{pmatrix} \tilde{b}_{11}(\mathbf{p}) & \tilde{b}_{21}(\mathbf{p})^H \\ \tilde{b}_{21}(\mathbf{p}) & \tilde{B}_{22}(\mathbf{p}) \end{pmatrix}, \tag{5}$$

where $\tilde{a}_{11}(\mathbf{p}), \tilde{b}_{11}(\mathbf{p}) \in \mathbb{R}$, $\tilde{a}_{11}(\mathbf{0}) = \sigma_1^2$, and $\tilde{b}_{11}(\mathbf{0}) = 1$. We now define the following vector-valued functions

$$\mathbf{f}(\mathbf{z}, \mathbf{w}, \mathbf{p}) = \tilde{a}_{21}(\mathbf{p}) + \tilde{A}_{22}(\mathbf{p})\mathbf{z} + \mathbf{w}\tilde{a}_{11}(\mathbf{p}) + \mathbf{w}\tilde{a}_{21}(\mathbf{p})^H\mathbf{z}, \tag{6}$$

$$\mathbf{g}(\mathbf{z}, \mathbf{w}, \mathbf{p}) = \tilde{b}_{21}(\mathbf{p}) + \tilde{B}_{22}(\mathbf{p})\mathbf{z} + \mathbf{w}\tilde{b}_{11}(\mathbf{p}) + \mathbf{w}\tilde{b}_{21}(\mathbf{p})^H\mathbf{z}, \tag{7}$$

where

$$\mathbf{f} = (f_1, \ldots, f_{n-1})^T, \quad \mathbf{g} = (g_1, \ldots, g_{n-1})^T,$$
$$\mathbf{z} = (\zeta_1, \ldots, \zeta_{n-1})^T \in \mathbb{C}^{n-1}, \quad \mathbf{w} = (\omega_1, \ldots, \omega_{n-1})^T \in \mathbb{C}^{n-1}, \quad \mathbf{p} \in \mathbb{R}^N.$$

Let

$$f_j = \varphi_j + i\psi_j, \quad \zeta_j = \xi_j + i\eta_j, \quad g_j = \tilde{\varphi}_j + i\tilde{\psi}_j,$$
$$\omega_j = \tilde{\xi}_j + i\tilde{\eta}_j, \quad j = 1, \ldots, n-1$$

and

$$\mathbf{x} = (\xi_1, \ldots, \xi_{n-1})^T \in \mathbb{R}^{n-1}, \quad \mathbf{y} = (\eta_1, \ldots, \eta_{n-1})^T \in \mathbb{R}^{n-1},$$
$$\tilde{\mathbf{x}} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_{n-1})^T \in \mathbb{R}^{n-1}, \quad \tilde{\mathbf{y}} = (\tilde{\eta}_1, \ldots, \tilde{\eta}_{n-1})^T \in \mathbb{R}^{n-1}.$$

It is clear that $\varphi_j(\mathbf{x}, \mathbf{y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \mathbf{p})$, $\psi_j(\mathbf{x}, \mathbf{y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \mathbf{p})$, $\tilde{\varphi}_j(\mathbf{x}, \mathbf{y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \mathbf{p})$ and $\tilde{\psi}_j(\mathbf{x}, \mathbf{y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \mathbf{p})$ are all real analytic functions of real variables $\mathbf{x}, \mathbf{y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}} \in \mathbb{R}^{n-1}$ and $\mathbf{p} \in \mathscr{B}(\mathbf{0})$, and

$$\varphi_j(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) = 0, \quad \psi_j(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) = 0, \quad \tilde{\varphi}_j(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) = 0, \quad \tilde{\psi}_j(\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) = 0 \tag{8}$$

for $j = 1, \ldots, n-1$. Notice that $f_1, \ldots, f_{n-1}$, $g_1, \ldots, g_{n-1}$ are all complex analytic functions of the complex variables $\zeta_1, \ldots, \zeta_{n-1}, \omega_1, \ldots, \omega_{n-1}$ for any $\mathbf{p} \in \mathscr{B}(\mathbf{0})$. Thus we have ([5, p.39, Theorem 8])

$$\begin{aligned} &\det \frac{\partial(\varphi_1, \ldots, \varphi_{n-1}, \psi_1, \ldots, \psi_{n-1}, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_{n-1}, \tilde{\psi}_1, \ldots, \tilde{\psi}_{n-1})}{\partial(\xi_1, \ldots, \xi_{n-1}, \eta_1, \ldots, \eta_{n-1}, \tilde{\xi}_1, \ldots, \tilde{\xi}_{n-1}, \tilde{\eta}_1, \ldots, \tilde{\eta}_{n-1})} \\ = &\left| \det \frac{\partial(f_1, \ldots, f_{n-1}, g_1, \ldots, g_{n-1})}{\partial(\zeta_1, \ldots, \zeta_{n-1}, \omega_1, \ldots, \omega_{n-1})} \right|^2. \end{aligned}$$

This, together with

$$\begin{aligned} &\det \left(\frac{\partial(f_1, \ldots, f_{n-1}, g_1, \ldots, g_{n-1})}{\partial(\zeta_1, \ldots, \zeta_{n-1}, \omega_1, \ldots, \omega_{n-1})} \right)_{\mathbf{z}=\mathbf{0},\mathbf{w}=\mathbf{0},\mathbf{p}=\mathbf{0}} = \det \begin{pmatrix} I_1 \otimes \tilde{A}_{22}(\mathbf{0}) & I_1 \otimes \tilde{B}_{22}(\mathbf{0}) \\ \tilde{a}_{11}(\mathbf{0}) \otimes I_{n-1} & \tilde{b}_{11}(\mathbf{0})^T \otimes I_{n-1} \end{pmatrix} \\ = \ &\det \begin{pmatrix} \Lambda_2^T \Lambda_2 & \Sigma_2^T \Sigma_2 \\ \sigma_1^2 I_{n-1} & I_{n-1} \end{pmatrix} = \det(\Lambda_2^T \Lambda_2 - \sigma_1^2 \Sigma_2^T \Sigma_2) \neq 0, \end{aligned}$$

implies that

$$\det \frac{\partial(\varphi_1, \ldots, \varphi_{n-1}, \psi_1, \ldots, \psi_{n-1}, \tilde{\varphi}_1, \ldots, \tilde{\varphi}_{n-1}, \tilde{\psi}_1, \ldots, \tilde{\psi}_{n-1})}{\partial(\xi_1, \ldots, \xi_{n-1}, \eta_1, \ldots, \eta_{n-1}, \tilde{\xi}_1, \ldots, \tilde{\xi}_{n-1}, \tilde{\eta}_1, \ldots, \tilde{\eta}_{n-1})} \neq 0,$$

where $\otimes$ denotes the Kronecker product (see for instance [9]). Therefore, by using the Implicit Function Theorem ([18, Theorem 1.2]), we know that the system of equations

$$\varphi_j(\mathbf{x},\mathbf{y},\tilde{\mathbf{x}},\tilde{\mathbf{y}},\mathbf{p})=\mathbf{0},\quad \psi_j(\mathbf{x},\mathbf{y},\tilde{\mathbf{x}},\tilde{\mathbf{y}},\mathbf{p})=\mathbf{0},\quad \tilde{\varphi}_j(\mathbf{x},\mathbf{y},\tilde{\mathbf{x}},\tilde{\mathbf{y}},\mathbf{p})=\mathbf{0},\quad \tilde{\psi}_j(\mathbf{x},\mathbf{y},\tilde{\mathbf{x}},\tilde{\mathbf{y}},\mathbf{p})=\mathbf{0}$$

for $j=1,\ldots,n-1$, i.e.,

$$\begin{cases} \mathbf{f}(\mathbf{z},\mathbf{w},\mathbf{p})=\mathbf{0} \\ \mathbf{g}(\mathbf{z},\mathbf{w},\mathbf{p})=\mathbf{0} \end{cases} \tag{9}$$

has a unique real analytic solution

$$\mathbf{x}=\mathbf{x}(\mathbf{p}),\quad \mathbf{y}=\mathbf{y}(\mathbf{p}),\quad \tilde{\mathbf{x}}=\tilde{\mathbf{x}}(\mathbf{p}),\quad \tilde{\mathbf{y}}=\tilde{\mathbf{y}}(\mathbf{p}),\quad \text{i.e.,}\quad \mathbf{z}=\mathbf{z}(\mathbf{p}),\quad \mathbf{w}=\mathbf{w}(\mathbf{p})$$

in some neighbourhood $\mathscr{B}_0\subset\mathscr{B}(\mathbf{0})$ of the origin, where

$$\mathbf{x}(\mathbf{0})=\mathbf{0},\quad \mathbf{y}(\mathbf{0})=\mathbf{0},\quad \tilde{\mathbf{x}}(\mathbf{0})=\mathbf{0},\quad \tilde{y}(\mathbf{0})=\mathbf{0},\quad \text{i.e.,}\quad \mathbf{z}(\mathbf{0})=\mathbf{0},\quad \mathbf{w}(\mathbf{0})=\mathbf{0}, \tag{10}$$

and

$$\det(I_{n-1}-\mathbf{w}(\mathbf{p})\mathbf{z}(\mathbf{p})^H)\neq 0\quad \forall \mathbf{p}\in\mathscr{B}_0. \tag{11}$$

Next, we construct a simple generalized singular value of $\{A(\mathbf{p}),B(\mathbf{p})\}$ and an associated generalized singular vector set. From (11), it follows that the matrix

$$\begin{pmatrix} 1 & \mathbf{w}(\mathbf{p})^H \\ \mathbf{z}(\mathbf{p}) & I_{n-1} \end{pmatrix}$$

is nonsingular for any $\mathbf{p}\in\mathscr{B}_0$. Thus we obtain by (4) and (5), for any $\mathbf{p}\in\mathscr{B}_0$,

$$\begin{pmatrix} 1 & \mathbf{w}(\mathbf{p})^H \\ \mathbf{z}(\mathbf{p}) & I_{n-1} \end{pmatrix}^H \tilde{A}(\mathbf{p}) \begin{pmatrix} 1 & \mathbf{w}(\mathbf{p})^H \\ \mathbf{z}(\mathbf{p}) & I_{n-1} \end{pmatrix} = \begin{pmatrix} a_1(\mathbf{p}) & 0 \\ 0 & A_2(\mathbf{p}) \end{pmatrix} \tag{12}$$

and

$$\begin{pmatrix} 1 & \mathbf{w}(\mathbf{p})^H \\ \mathbf{z}(\mathbf{p}) & I_{n-1} \end{pmatrix}^H \tilde{B}(\mathbf{p}) \begin{pmatrix} 1 & \mathbf{w}(\mathbf{p})^H \\ \mathbf{z}(\mathbf{p}) & I_{n-1} \end{pmatrix} = \begin{pmatrix} b_1(\mathbf{p}) & 0 \\ 0 & B_2(\mathbf{p}) \end{pmatrix}, \tag{13}$$

where

$$\begin{aligned} a_1(\mathbf{p}) &= (\mathbf{q}_1+Q_2\mathbf{z}(\mathbf{p}))^H A(\mathbf{p})^H A(\mathbf{p})(\mathbf{q}_1+Q_2\mathbf{z}(\mathbf{p})) \\ &= \tilde{a}_{11}(\mathbf{p})+\mathbf{z}(\mathbf{p})^H\tilde{a}_{21}(\mathbf{p})+\tilde{a}_{21}(\mathbf{p})^H\mathbf{z}(\mathbf{p})+\mathbf{z}(\mathbf{p})^H\tilde{A}_{22}(\mathbf{p})\mathbf{z}(\mathbf{p}), \end{aligned} \tag{14}$$

and

$$\begin{aligned} b_1(\mathbf{p}) &= (\mathbf{q}_1+Q_2\mathbf{z}(\mathbf{p}))^H B(\mathbf{p})^H B(\mathbf{p})(\mathbf{q}_1+Q_2\mathbf{z}(\mathbf{p})) \\ &= \tilde{b}_{11}(\mathbf{p})+\mathbf{z}(\mathbf{p})^H\tilde{b}_{21}(\mathbf{p})+\tilde{b}_{21}(\mathbf{p})^H\mathbf{z}(\mathbf{p})+\mathbf{z}(\mathbf{p})^H\tilde{B}_{22}(\mathbf{p})\mathbf{z}(\mathbf{p}) \end{aligned} \tag{15}$$

with $a_1(\mathbf{0})=\sigma_1^2$ and $b_1(\mathbf{0})=1$. We observe that for sufficiently small $\mathscr{B}_0$,

$$a_1(\mathbf{p})>0,\quad b_1(\mathbf{p})>0\quad \forall\mathbf{p}\in\mathscr{B}_0.$$

Hence, we can define a positive valued function $\sigma_1:\mathscr{B}_0\to R$ by

$$\sigma_1(\mathbf{p})=a_1(\mathbf{p})^{\frac{1}{2}}b_1(\mathbf{p})^{-\frac{1}{2}}\quad \forall\mathbf{p}\in\mathscr{B}_0. \tag{16}$$

In addition, let

$$\mathbf{q}_1(\mathbf{p}) = Q \begin{pmatrix} 1 \\ \mathbf{z}(\mathbf{p}) \end{pmatrix} b_1(\mathbf{p})^{-\frac{1}{2}} \quad \forall \mathbf{p} \in \mathscr{B}_0, \tag{17}$$

$$\mathbf{u}_1(\mathbf{p}) = A(\mathbf{p})\mathbf{q}_1(\mathbf{p})/\sigma_1(\mathbf{p}), \quad \mathbf{v}_1(\mathbf{p}) = B(\mathbf{p})\mathbf{q}_1(\mathbf{p}) \quad \forall \mathbf{p} \in \mathscr{B}_0. \tag{18}$$

By (4)–(18), it is easy to know that the functions $\mathbf{u}_1(\mathbf{p})$, $\mathbf{v}_1(\mathbf{p})$, and $\mathbf{q}_1(\mathbf{p})$ are such that $\mathrm{Re}[\mathbf{u}_1(\mathbf{p})]$, $\mathrm{Im}[\mathbf{u}_1(\mathbf{p})]$, $\mathrm{Re}[\mathbf{v}_1(\mathbf{p})]$, $\mathrm{Im}[\mathbf{v}_1(\mathbf{p})]$, $\mathrm{Re}[\mathbf{q}_1(\mathbf{p})]$, and $\mathrm{Im}[\mathbf{q}_1(\mathbf{p})]$ are all real analytic in $\mathscr{B}_o$ with

$$A(\mathbf{p})\mathbf{q}_1(\mathbf{p}) = \sigma_1(\mathbf{p})\mathbf{u}_1(\mathbf{p}), \quad B(\mathbf{p})\mathbf{q}_1(\mathbf{p}) = \mathbf{v}_1(\mathbf{p}), \quad \mathbf{u}_1(\mathbf{p})^H\mathbf{u}_1(\mathbf{p}) = \mathbf{v}_1(\mathbf{p})^H\mathbf{v}_1(\mathbf{p}) = 1 \tag{19}$$

and

$$\mathbf{u}_1(\mathbf{0}) = \mathbf{u}_1, \quad \mathbf{v}_1(\mathbf{0}) = \mathbf{v}_1, \quad \mathbf{q}_1(\mathbf{0}) = \mathbf{q}_1. \tag{20}$$

By using the perturbation theorem for generalized singular values (see for instance [16]), it is easy to see that, if the neighborhood $\mathscr{B}_0$ is small enough, the generalized singular value $(\sigma_1(\mathbf{p}), 1)$ of $\{A(\mathbf{p}), B(\mathbf{p})\}$ such that $(\sigma_1(\mathbf{p}), 1)$ is simple and $\sigma_1(\mathbf{p})$ is a real analytic function of $\mathbf{p}$ in $\mathscr{B}_0$ and $\sigma_1(\mathbf{0}) = \sigma_1$. □

Theorem 3.2 *Under the same assumptions as in Theorem 3.1, the following formulas for the simple generalized singular value* $(\sigma_1(\mathbf{p}), 1)$ *and the generalized singular vector set* $\{\mathbf{u}_1(\mathbf{p}), \mathbf{v}_1(\mathbf{p}), \mathbf{q}_1(\mathbf{p})\}$ *defined in* (14)–(18) *hold:*

$$\left(\frac{\partial \sigma_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} = \mathrm{Re}\left[\mathbf{u}_1^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] - \sigma_1 \mathrm{Re}\left[\mathbf{v}_1^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right], \tag{21}$$

$$\begin{aligned} \left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} &= Q_2\Phi_1Q_2^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)^H_{\mathbf{p}=\mathbf{0}} \mathbf{u}_1 + Q_2\Phi_2U_2^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1 \\ &\quad - Q_2\Phi_3Q_2^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)^H_{\mathbf{p}=\mathbf{0}} \mathbf{v}_1 - Q_2\Phi_4V_2^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1 \\ &\quad - \mathrm{Re}\left[\mathbf{v}_1^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] \mathbf{q}_1, \end{aligned} \tag{22}$$

$$\begin{aligned} \left(\frac{\partial \mathbf{u}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} &= \frac{1}{\sigma_1}\left\{\left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1 + A(\mathbf{0})\left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} - \mathbf{u}_1\left(\frac{\partial \sigma_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}\right\} \\ &= \frac{1}{\sigma_1}\left\{U_2\Lambda_2\Phi_1Q_2^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)^H_{\mathbf{p}=\mathbf{0}} \mathbf{u}_1 + U_2\Lambda_2\Phi_2U_2^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right\} \\ &\quad - \frac{1}{\sigma_1}\left\{U_2\Lambda_2\Phi_3Q_2^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)^H_{\mathbf{p}=\mathbf{0}} \mathbf{v}_1 + U_2\Lambda_2\Phi_4V_2^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right\} \\ &\quad + \frac{1}{\sigma_1}\left\{\left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1 - \mathrm{Re}\left[\mathbf{u}_1^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right]\mathbf{u}_1\right\}, \end{aligned} \tag{23}$$

$$\begin{aligned}
\left(\frac{\partial \mathbf{v}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} &= B(\mathbf{0})\left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} + \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1 \\
&= V_2\Sigma_2\Phi_1 Q_2^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)^H_{\mathbf{p}=\mathbf{0}} \mathbf{u}_1 + V_2\Sigma_2\Phi_2 U_2^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1 \\
&\quad -V_2\Sigma_2\Phi_3 Q_2^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)^H_{\mathbf{p}=\mathbf{0}} \mathbf{v}_1 - V_2\Sigma_2\Phi_4 V_2^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1 \\
&\quad + \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1 - \mathrm{Re}\Big[\mathbf{v}_1^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\Big]\mathbf{v}_1, \qquad (24)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial^2 \sigma_1(\mathbf{p})}{\partial p_j \partial p_k}\right)_{\mathbf{p}=\mathbf{0}} &= \mathrm{Re}\left[\mathbf{u}_1^H \left(\frac{\partial^2 A(\mathbf{p})}{\partial p_j \partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] - \sigma_1 \mathrm{Re}\left[\mathbf{v}_1^H \left(\frac{\partial^2 B(\mathbf{p})}{\partial p_j \partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] \\
&\quad + \frac{1}{\sigma_1}\mathrm{Re}\left[\mathbf{q}_1^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)^H_{\mathbf{p}=\mathbf{0}} \left(\frac{\partial A(\mathbf{p})}{\partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] \\
&\quad - \sigma_1\mathrm{Re}\left[\mathbf{q}_1^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)^H_{\mathbf{p}=\mathbf{0}} \left(\frac{\partial B(\mathbf{p})}{\partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] \\
&\quad + \mathrm{Re}\left[\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{q}_1 \end{pmatrix}^H D_k^H C_0 D_j \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{q}_1 \end{pmatrix}\right] + \sigma_1 \mathrm{Re}\left[\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{q}_1 \end{pmatrix}^H S_j^H C_2 S_k \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{q}_1 \end{pmatrix}\right] \\
&\quad - \sigma_1 \mathrm{Re}\left[\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{q}_1 \end{pmatrix}^H (S_j^H C_1 D_k + S_k^H C_1 D_j) \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{q}_1 \end{pmatrix}\right] \\
&\quad - \frac{1}{\sigma_1}\mathrm{Re}\left[\mathbf{u}_1^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] \mathrm{Re}\left[\mathbf{u}_1^H \left(\frac{\partial A(\mathbf{p})}{\partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] \\
&\quad - \mathrm{Re}\left[\mathbf{u}_1^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] \mathrm{Re}\left[\mathbf{v}_1^H \left(\frac{\partial B(\mathbf{p})}{\partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] \\
&\quad - \mathrm{Re}\left[\mathbf{u}_1^H \left(\frac{\partial A(\mathbf{p})}{\partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] \mathrm{Re}\left[\mathbf{v}_1^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] \\
&\quad + 3\sigma_1 \mathrm{Re}\left[\mathbf{v}_1^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] \mathrm{Re}\left[\mathbf{v}_1^H \left(\frac{\partial B(\mathbf{p})}{\partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] \qquad (25)
\end{aligned}$$

for $j, k = 1, \ldots, N$, *where*

$$\Phi = (\sigma_1^2 \Sigma_2^T \Sigma_2 - \Lambda_2^T \Lambda_2)^{-1},$$

$$\Phi_1 = \sigma_1 \Phi, \quad \Phi_2 = \Phi \Lambda_2^T, \quad \Phi_3 = \sigma_1^2 \Phi, \quad \Phi_4 = \sigma_1^2 \Phi \Sigma_2^T,$$

$$D_j = \begin{pmatrix} \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)^H_{\mathbf{p}=\mathbf{0}} & 0 \\ 0 & \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \end{pmatrix}, \quad S_j = \begin{pmatrix} \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)^H_{\mathbf{p}=\mathbf{0}} & 0 \\ 0 & \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \end{pmatrix},$$

$$C_0 = \begin{pmatrix} Q_2\Phi_1 Q_2^H & Q_2\Phi_2 U_2^H \\ U_2\Phi_2^H Q_2^H & \frac{1}{\sigma_1} U_2\Lambda_2\Phi_2 U_2^H \end{pmatrix}, \quad C_1 = \begin{pmatrix} Q_2\Phi_1 Q_2^H & Q_2\Phi_2 U_2^H \\ V_2\Sigma_2\Phi_1 Q_2^H & V_2\Sigma_2\Phi_2 U_2^H \end{pmatrix},$$

$$C_2 = \begin{pmatrix} Q_2\Phi_3 Q_2^H & Q_2\Phi_4 V_2^H \\ V_2\Sigma_2\Phi_3 Q_2^H & V_2\Sigma_2\Phi_4 V_2^H \end{pmatrix}.$$

Proof: 1) By Theorem 3.1 (see (19) and (20)), we have

$$\sigma_1(\mathbf{p}) = \mathbf{u}_1(\mathbf{p})^H A(\mathbf{p})\mathbf{q}_1(\mathbf{p}) = \mathbf{q}_1(\mathbf{p})^H A(\mathbf{p})^H \mathbf{u}_1(\mathbf{p}), \tag{26}$$

$$1 = \mathbf{v}_1(\mathbf{p})^H B(\mathbf{p})\mathbf{q}_1(\mathbf{p}) = \mathbf{q}_1(\mathbf{p})^H B(\mathbf{p})^H \mathbf{v}_1(\mathbf{p}). \tag{27}$$

It follows from (26) and (19) that

$$\frac{\partial \sigma_1(\mathbf{p})}{\partial p_j} = \sigma_1(\mathbf{p}) \left(\frac{\partial \mathbf{u}_1(\mathbf{p})}{\partial p_j}\right)^H \mathbf{u}_1(\mathbf{p}) + \mathbf{u}_1(\mathbf{p})^H \frac{\partial A(\mathbf{p})}{\partial p_j}\mathbf{q}_1(\mathbf{p}) + \mathbf{u}_1(\mathbf{p})^H A(\mathbf{p}) \left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j}\right) \tag{28}$$

and

$$\frac{\partial \sigma_1(\mathbf{p})}{\partial p_j} = \left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j}\right)^H A(\mathbf{p})^H \mathbf{u}_1(\mathbf{p}) + \mathbf{q}_1(\mathbf{p})^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)^H \mathbf{u}_1(\mathbf{p}) + \sigma_1(\mathbf{p})\mathbf{u}_1(\mathbf{p})^H \left(\frac{\partial \mathbf{u}_1(\mathbf{p})}{\partial p_j}\right). \tag{29}$$

By(27) and (19), we get

$$0 = \left(\frac{\partial \mathbf{v}_1(\mathbf{p})}{\partial p_j}\right)^H \mathbf{v}_1(\mathbf{p}) + \mathbf{v}_1(\mathbf{p})^H \frac{\partial B(\mathbf{p})}{\partial p_j}\mathbf{q}_1(\mathbf{p}) + \mathbf{v}_1(\mathbf{p})^H B(\mathbf{p}) \left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j}\right) \tag{30}$$

and

$$0 = \left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j}\right)^H B(\mathbf{p})^H \mathbf{v}_1(\mathbf{p}) + \mathbf{q}_1(\mathbf{p})^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)^H \mathbf{v}_1(\mathbf{p}) + \mathbf{v}_1(\mathbf{p})^H \left(\frac{\partial \mathbf{v}_1(\mathbf{p})}{\partial p_j}\right). \tag{31}$$

Using (28)–(31) and $\mathbf{u}_1(\mathbf{p})^H\mathbf{u}_1(\mathbf{p}) = \mathbf{v}_1(\mathbf{p})^H\mathbf{v}_1(\mathbf{p}) = 1$, we have

$$\begin{aligned} \frac{\partial \sigma_1(\mathbf{p})}{\partial p_j} &= \frac{1}{2}\left[\mathbf{u}_1(\mathbf{p})^H \frac{\partial A(\mathbf{p})}{\partial p_j}\mathbf{q}_1(\mathbf{p}) + \mathbf{q}_1(\mathbf{p})^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)^H \mathbf{u}_1(\mathbf{p})\right] \\ &\quad -\frac{1}{2}\sigma_1(\mathbf{p})\left[\mathbf{v}_1(\mathbf{p})^H \frac{\partial B(\mathbf{p})}{\partial p_j}\mathbf{q}_1(\mathbf{p}) + \mathbf{q}_1(\mathbf{p})^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)^H \mathbf{v}_1(\mathbf{p})\right] \\ &\quad +\frac{1}{2}\left(\mathbf{u}_1(\mathbf{p})^H A(\mathbf{p}) - \sigma_1(\mathbf{p})\mathbf{v}_1(\mathbf{p})^H B(\mathbf{p})\right)\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j} \\ &\quad +\frac{1}{2}\left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j}\right)^H \left(A(\mathbf{p})^H\mathbf{u}_1(\mathbf{p}) - \sigma_1(\mathbf{p})B(\mathbf{p})^H\mathbf{v}_1(\mathbf{p})\right). \end{aligned} \tag{32}$$

Substituting $\mathbf{p} = \mathbf{0}$ into (32) and using $\mathbf{u}_1^H A(\mathbf{0}) - \sigma_1 \mathbf{v}_1^H B(\mathbf{0}) = 0_{1\times n}$, we obtain (21).

2) By Theorem 3.1 (see (4), (5), (12), (13), and (17)), we get

$$A(\mathbf{p})^H A(\mathbf{p})\mathbf{q}_1(\mathbf{p}) = \sigma_1^2(\mathbf{p}) B(\mathbf{p})^H B(\mathbf{p})\mathbf{q}_1(\mathbf{p}),$$

which yields

$$
\begin{aligned}
&\left(\sigma_1^2 B(\mathbf{0})^H B(\mathbf{0}) - A(\mathbf{0})^H A(\mathbf{0})\right) \left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \\
&= \left[\left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}^H A(\mathbf{0}) + A(\mathbf{0})^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}\right] \mathbf{q}_1 - 2\sigma_1 \left(\frac{\partial \sigma_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} B(\mathbf{0})^H B(\mathbf{0}) \mathbf{q}_1 \\
&-\sigma_1^2 \left[\left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}^H B(\mathbf{0}) + B(\mathbf{0})^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}\right] \mathbf{q}_1. \qquad (33)
\end{aligned}
$$

This, together with (3), (10), and (17), gives rise to

$$
\begin{aligned}
&\begin{pmatrix} 0 & 0 \\ 0 & \sigma_1^2 \Sigma_2^H \Sigma_2 - \Lambda_2^H \Lambda_2 \end{pmatrix} \begin{pmatrix} 0 \\ \left(\frac{\partial \mathbf{z}(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \end{pmatrix} \\
= \; & \sigma_1 Q^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}^H \mathbf{u}_1 + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \Lambda_2^H \end{pmatrix} U^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1 - 2\sigma_1 \left(\frac{\partial \sigma_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&-\sigma_1^2 \left\{ Q^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}^H \mathbf{v}_1 + \begin{pmatrix} 1 & 0 \\ 0 & \Sigma_2^H \end{pmatrix} V^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1 \right\}
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\left(\frac{\partial \mathbf{z}(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \\
= \; & (\sigma_1^2 \Sigma_2^T \Sigma_2 - \Lambda_2^T \Lambda_2)^{-1} \left[\sigma_1 Q_2^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}^H \mathbf{u}_1 + \Lambda_2^H U_2^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] \\
&-\sigma_1^2 (\sigma_1^2 \Sigma_2^T \Sigma_2 - \Lambda_2^T \Lambda_2)^{-1} \left[Q_2^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}^H \mathbf{v}_1 + \Sigma_2^H V_2^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right]. \qquad (34)
\end{aligned}
$$

Moreover, we have by (15) and (3),

$$
\left(\frac{\partial b_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} = 2\mathrm{Re}\left[\mathbf{v}_1^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] \qquad (35)
$$

and using (17), we get

$$
\left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} = Q_2 \left(\frac{\partial \mathbf{z}(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} - \frac{1}{2}\mathbf{q}_1 \left(\frac{\partial b_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}. \qquad (36)
$$

Combining (34), (35), and (36) yields (22).

3) From (19), we obtain

$$
\left(\frac{\partial \mathbf{u}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} = \frac{1}{\sigma_1}\left[\left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1 + A(\mathbf{0}) \left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} - \mathbf{u}_1 \left(\frac{\partial \sigma_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}\right].
$$

This, together with (21), (22), and the relation $A(\mathbf{0})Q_2 = U_2\Lambda_2$, yields (23).

4) By using (19) again, we obtain

$$\left(\frac{\partial \mathbf{v}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} = B(\mathbf{0})\left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} + \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1, \tag{37}$$

which, together with (22) and the relation $B(\mathbf{0})Q_2 = V_2\Sigma_2$, gives rise to (24).

5) By (32), we have

$$\begin{aligned}
&\left(\frac{\partial^2 \sigma_1(\mathbf{p})}{\partial p_j \partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \\
= \;& \mathrm{Re}\left[\left(\frac{\partial \mathbf{u}_1(\mathbf{p})}{\partial p_k}\right)^H_{\mathbf{p}=\mathbf{0}} \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] + \mathrm{Re}\left[\mathbf{u}_1^H \left(\frac{\partial^2 A(\mathbf{p})}{\partial p_j \partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] \\
& + \mathrm{Re}\left[\mathbf{u}_1^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_k}\right)_{\mathbf{p}=\mathbf{0}}\right] - \left(\frac{\partial \sigma_1(\mathbf{p})}{\partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \mathrm{Re}\left[\mathbf{v}_1^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] \\
& - \sigma_1 \mathrm{Re}\left[\left(\frac{\partial \mathbf{v}_1(\mathbf{p})}{\partial p_k}\right)^H_{\mathbf{p}=\mathbf{0}} \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] - \sigma_1 \mathrm{Re}\left[\mathbf{v}_1^H \left(\frac{\partial^2 B(\mathbf{p})}{\partial p_j \partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{q}_1\right] \\
& - \sigma_1 \mathrm{Re}\left[\mathbf{v}_1^H \left(\frac{\partial B(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_k}\right)_{\mathbf{p}=\mathbf{0}}\right] \\
& + \mathrm{Re}\left[\left(\frac{\partial \mathbf{u}_1(\mathbf{p})}{\partial p_k}\right)^H_{\mathbf{p}=\mathbf{0}} A(\mathbf{0}) \left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}\right] + \mathrm{Re}\left[\mathbf{u}_1^H \left(\frac{\partial A(\mathbf{p})}{\partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}\right] \\
& - \left(\frac{\partial \sigma_1(\mathbf{p})}{\partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \mathrm{Re}\left[\mathbf{v}_1^H B(\mathbf{0}) \left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}\right] \\
& - \sigma_1 \mathrm{Re}\left[\left(\frac{\partial \mathbf{v}_1(\mathbf{p})}{\partial p_k}\right)^H_{\mathbf{p}=\mathbf{0}} B(\mathbf{0}) \left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}\right] - \sigma_1 \mathrm{Re}\left[\mathbf{v}_1^H \left(\frac{\partial B(\mathbf{p})}{\partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \left(\frac{\partial \mathbf{q}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}\right].
\end{aligned}$$

Combining it with (21), (22), (23), and (24) yields (25). □

We remark that as in [7, 21, 22], we derive our sensitivity results based on the Implicit Function Theorem and the definition of the generalized singular value decomposition. Also, if $A(\mathbf{p}) \in \mathbb{R}^{m\times n}$, $B(\mathbf{p}) \in \mathbb{R}^{l\times n}$, the first order partial derivatives related to the simple generalized singular value $(\sigma_1(\mathbf{p}), 1)$ and the generalized singular vector set $\{\mathbf{u}_1(\mathbf{p}), \mathbf{v}_1(\mathbf{p}), \mathbf{q}_1(\mathbf{p})\}$ of $\{A(\mathbf{p}), B(\mathbf{p})\}$ in Theorem 3.2 are same as in [7, Theorem 2.1] under the assumption that the generalized singular value $(\sigma_1, 1)$ of $\{A(\mathbf{0}), B(\mathbf{0})\}$ is simple.

On the other hand, suppose that $m \geq n$. Let $B(\mathbf{p}) = I_n$ for all $\mathbf{p} \in \mathbb{R}^N$. By (3), there exist two unitary matrices $U = (\mathbf{u}_1, U_2) \in \mathbb{C}^{m\times m}, V = (\mathbf{v}_1, V_2) \in \mathbb{C}^{n\times n}$, and a nonsingular matrix $Q = V \in \mathbb{C}^{n\times n}$ such that

$$U^T A(\mathbf{0}) V = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix} \quad \text{and} \quad V^H B(\mathbf{0}) Q = V^H V = \begin{pmatrix} 1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}, \tag{38}$$

$$\Lambda_2 = \begin{pmatrix} \sigma_2 & & \\ & \ddots & \\ & & \sigma_n \\ & 0 & \end{pmatrix} \in \mathbb{R}^{(m-1)\times(n-1)}, \quad \Sigma_2 = I_{n-1},$$

where $\sigma_2, \ldots, \sigma_n \geq 0$ and $0 < \sigma_1 \neq \sigma_j$ for $j = 2, \ldots, n$. That is, σ_1 is a simple nonzero singular value of $A(\mathbf{0})$, $\mathbf{v}_1 \in \mathbb{C}^n$ and $\mathbf{u}_1 \in \mathbb{C}^m$ are associated unit right and unit left singular vectors, respectively.

By using the unitarity of U, we have

$$\begin{aligned} \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{v}_1 &= UU^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{v}_1 \\ &= \left\{\mathbf{u}_1^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{v}_1\right\} \mathbf{u}_1 + U_2 U_2^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{v}_1. \end{aligned} \tag{39}$$

Also, it is easy to check that

$$I_{m-1} + \Lambda_2(\sigma_1^2 I_{n-1} - \Lambda_2^T \Lambda_2)^{-1} \Lambda_2^T = \sigma_1^2(\sigma_1^2 I_{m-1} - \Lambda_2 \Lambda_2^T)^{-1}. \tag{40}$$

Therefore, by using Theorems 3.1 and 3.2, (39), and (40), it is easy to derive the same results as in [21] on the analyticity of simple nonzero singular values of a matrix analytically dependent on several parameters.

Corollary 3.3 *Let $\mathbf{p} \in \mathbb{R}^N$ and $A(\mathbf{p}) \in \mathbb{C}^{m\times n}$ $(m \geq n)$. Suppose that $\mathrm{Re}[A(\mathbf{p})]$ and $\mathrm{Im}[A(\mathbf{p})]$ are real analytic matrix-valued functions of $\mathbf{p}$ in some neighborhood $\mathscr{B}(\mathbf{0})$ of the origin. If $\sigma_1 > 0$ is a simple nonzero singular value of $A(\mathbf{0})$ and there exist two unit vectors $\mathbf{u}_1 \in \mathbb{C}^m$ and $\mathbf{v}_1 \in \mathbb{C}^n$ such that $\mathbf{u}_1$ and $\mathbf{v}_1$ are associated left and right singular vectors, respectively, i.e., there exist two unitary matrices $U = (\mathbf{u}_1, U_2) \in \mathbb{C}^{m\times m}$ and $V = (\mathbf{v}_1, V_2) \in \mathbb{C}^{n\times n}$ such that the first equality of (38) holds, then*

1) *there exists a simple singular value $\sigma_1(\mathbf{p}) > 0$ of $A(\mathbf{p})$ such that $\sigma_1(\mathbf{p})$ is a real analytic function of $\mathbf{p}$ in some neighborhood $\mathscr{B}_0 \subset \mathscr{B}(\mathbf{0})$ of the origin and $\sigma_1(\mathbf{0}) = \sigma_1$.*

2) *there exist unit vectors $\mathbf{u}_1(\mathbf{p}) \in \mathbb{C}^m$ and $\mathbf{v}_1(\mathbf{p}) \in \mathbb{C}^n$ such that $\mathrm{Re}[\mathbf{u}_1(\mathbf{p})]$, $\mathrm{Im}[\mathbf{u}_1(\mathbf{p})]$, $\mathrm{Re}[\mathbf{v}_1(\mathbf{p})]$, and $\mathrm{Im}[\mathbf{v}_1(\mathbf{p})]$ are all real analytic functions of $\mathbf{p}$ in $\mathscr{B}_0$ and $\mathbf{u}_1(\mathbf{p})$ and $\mathbf{v}_1(\mathbf{p})$ are the left and right singular vectors of $A(\mathbf{p})$ corresponding to the simple singular value $\sigma_1(\mathbf{p})$, i.e.,*

   $$A(\mathbf{p})\mathbf{v}_1(\mathbf{p}) = \sigma_1(\mathbf{p})\mathbf{u}_1(\mathbf{p}) \quad \textit{and} \quad A(\mathbf{p})^H \mathbf{u}_1(\mathbf{p}) = \sigma_1(\mathbf{p})\mathbf{v}_1(\mathbf{p}),$$

   where $\mathbf{u}_1(\mathbf{0}) = \mathbf{u}_1$ and $\mathbf{v}_1(\mathbf{0}) = \mathbf{v}_1$.

Moreover, the simple nonzero singular value $\sigma_1(\mathbf{p})$ is given by

$$\sigma_1(\mathbf{p}) = \left(\tilde{a}_{11}(\mathbf{p}) + \mathbf{z}(\mathbf{p})^H \tilde{a}_{21}(\mathbf{p}) + \tilde{a}_{21}(\mathbf{p})^H \mathbf{z}(\mathbf{p}) + \mathbf{z}(\mathbf{p})^H \tilde{A}_{22}(\mathbf{p})\mathbf{z}(\mathbf{p})\right)^{\frac{1}{2}} \left(1 + \mathbf{z}(\mathbf{p})^H \mathbf{z}(\mathbf{p})\right)^{-\frac{1}{2}} \tag{41}$$

for all $\mathbf{p} \in \mathscr{B}_0$*, where* $\tilde{a}_{11}(\mathbf{p})$*,* $\tilde{a}_{z1}(\mathbf{p})$*,* $\tilde{A}_{22}(\mathbf{p})$ *are given by*

$$V^H A(\mathbf{p})^H A(\mathbf{p}) V := \begin{pmatrix} \tilde{a}_{11}(\mathbf{p}) & \tilde{a}_{21}(\mathbf{p})^H \\ \tilde{a}_{21}(\mathbf{p}) & \tilde{A}_{22}(\mathbf{p}) \end{pmatrix} \quad \forall \mathbf{p} \in \mathscr{B}_0$$

and $\mathbf{z}(\mathbf{p}) \in \mathbb{R}^{n-1}$ *is the real analytic solution of* $\mathbf{f}(\mathbf{z}, -\mathbf{z}, \mathbf{p}) = \mathbf{0}$ *in* $\mathscr{B}_0$*, where* $\mathbf{f}(\mathbf{z}, \mathbf{w}, \mathbf{p})$ *is defined in* (6)*,* $\mathbf{z}(\mathbf{0}) = \mathbf{0}$ *and* $det(I_{n-1} + \mathbf{z}(\mathbf{p})\mathbf{z}(\mathbf{p})^H) \neq 0$ *for all* $\mathbf{p} \in \mathscr{B}_0$*. In addition, the associated unit singular vectors* $\mathbf{v}_1(\mathbf{p})$ *and* $\mathbf{u}_1(\mathbf{p})$ *are given by*

$$\mathbf{v}_1(\mathbf{p}) = V \begin{pmatrix} 1 \\ \mathbf{z}(\mathbf{p}) \end{pmatrix} \left(1 + \mathbf{z}(\mathbf{p})^H \mathbf{z}(\mathbf{p})\right)^{-\frac{1}{2}}, \quad \mathbf{u}_1(\mathbf{p}) = A(\mathbf{p})\mathbf{v}_1(\mathbf{p})/\sigma_1(\mathbf{p}) \quad \forall \mathbf{p} \in \mathscr{B}_0. \tag{42}$$

Finally, the following formulas for the simple nonzero singular value $\sigma_1(\mathbf{p})$ *and the associated singular vectors* $\mathbf{v}_1(\mathbf{p})$ *and* $\mathbf{u}_1(\mathbf{p})$ *defined by* (41)*–*(42) *hold:*

$$\left(\frac{\partial \sigma_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} = \mathrm{Re}\left[\mathbf{u}_1^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{v}_1\right], \tag{43}$$

$$\left(\frac{\partial \mathbf{v}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} = V_2 \left(\Phi_1 V_2^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)^H_{\mathbf{p}=\mathbf{0}}, \quad \Phi_2^T U_2^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}\right) \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{v}_1 \end{pmatrix} \tag{44}$$

$$\begin{aligned} \left(\frac{\partial \mathbf{u}_1(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} &= U_2 \left(\Phi_2 V_2^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)^H_{\mathbf{p}=\mathbf{0}}, \quad \Phi_3 U_2^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}}\right) \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{v}_1 \end{pmatrix} \\ &+ \frac{i}{\sigma_1} \mathrm{Im}\left[\mathbf{u}_1^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{v}_1\right] \mathbf{u}_1, \end{aligned} \tag{45}$$

$$\begin{aligned} \left(\frac{\partial^2 \sigma_1(\mathbf{p})}{\partial p_j \partial p_k}\right)_{\mathbf{p}=\mathbf{0}} &= \mathrm{Re}\left[\mathbf{u}_1^H \left(\frac{\partial^2 A(\mathbf{p})}{\partial p_j \partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{v}_1\right] \\ &+ \mathrm{Re}\left[\begin{pmatrix} \mathbf{u}_1 \\ \mathbf{v}_1 \end{pmatrix}^H \begin{pmatrix} \left(\frac{\partial A(\mathbf{p})}{\partial p_k}\right)^H_{\mathbf{p}=\mathbf{0}} & 0 \\ 0 & \left(\frac{\partial A(\mathbf{p})}{\partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \end{pmatrix}^H \right. \\ &\times \left. \begin{pmatrix} V_2 \Phi_1 V_2^H & V_2 \Phi_2^T U_2^H \\ U_2 \Phi_2 V_2^H & U_2 \Phi_3 U_2^H \end{pmatrix} \begin{pmatrix} \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)^H_{\mathbf{p}=\mathbf{0}} & 0 \\ 0 & \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{v}_1 \end{pmatrix}\right] \\ &+ \frac{1}{\sigma_1} \mathrm{Im}\left[\mathbf{u}_1^H \left(\frac{\partial A(\mathbf{p})}{\partial p_j}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{v}_1\right] \mathrm{Im}\left[\mathbf{u}_1^H \left(\frac{\partial A(\mathbf{p})}{\partial p_k}\right)_{\mathbf{p}=\mathbf{0}} \mathbf{v}_1\right], \end{aligned} \tag{46}$$

where $j, k = 1, \ldots, N$*,* $\mathbf{u}_1, \mathbf{v}_1, U_2$ *and* V_2 *are defined by* (38)*, and*

$$\Phi_1 = \sigma_1(\sigma_1^2 I_{n-1} - \Lambda_2^T \Lambda_2)^{-1}, \quad \Phi_2 = \Lambda_2(\sigma_1^2 I_{n-1} - \Lambda_2^T \Lambda_2)^{-1}, \quad \Phi_3 = \sigma_1(\sigma_1^2 I_{m-1} - \Lambda_2 \Lambda_2^T)^{-1}$$

in which Λ_2 *is defined by* (38)*.*

4 Applications

In this section, we give some examples to show that our results are useful for computing the sensitivity and the second order perturbation expansions of simple nonzero finite generalized singular values of a complex matrix pair analytically dependent on several parameters. Based on Theorems 3.1 and 3.2, we first define the sensitivity of simple nonzero finite generalized singular values as follows [7].

Definition 4.1 *Let* $\mathbf{p} = (p_1, \ldots, p_N)^T \in \mathbb{R}^N$, $A(\mathbf{p}) \in \mathbb{C}^{m\times n}$, *and* $B(\mathbf{p}) \in \mathbb{C}^{l\times n}$. *Suppose that* $(A(\mathbf{p})^H,\ B(\mathbf{p})^H)$ *has full row rank and* $\mathrm{Re}[A(\mathbf{p})]$, $\mathrm{Re}[B(\mathbf{p})]$, $\mathrm{Im}[A(\mathbf{p})]$ *and* $\mathrm{Im}[A(\mathbf{p})]$ *are real analytic matrix-valued functions of* $\mathbf{p}$ *in some neighborhood* $\mathscr{B}(\mathbf{0})$ *of the origin. If* $\{A(\mathbf{0}), B(\mathbf{0})\}$ *has the GSVD* (3)*, then the quantity*

$$s_{p_j}(\sigma_1) = \left| \left(\frac{\partial \sigma_1(\mathbf{p})}{\partial p_j} \right)_{\mathbf{p}=\mathbf{0}} \right| \tag{47}$$

is called the sensitivity of the simple nonzero finite generalized singular value $(\sigma_1, 1)$ *with respect to the parameter* p_j*, the quantity*

$$s_{p_{i_1}, p_{i_2}, \ldots, p_{i_m}}(\sigma_1) = \sqrt{\sum_{k=1}^{m} s^2_{p_{i_k}}(\sigma_1)} \tag{48}$$

is called the sensitivity of the simple nonzero finite generalized singular value $(\sigma_1, 1)$ *with respect to the parameter* $p_{i_1}, p_{i_2}, \ldots, p_{i_m}$*, the quantity*

$$s_p(\sigma_1) = \sqrt{\sum_{j=1}^{N} s^2_{p_j}(\sigma_1)} \tag{49}$$

is called the sensitivity of the simple nonzero finite generalized singular value $(\sigma_1, 1)$.

Example 4.2 *Let*

$$A(\mathbf{p}) = \begin{pmatrix} 3 + 2p_1 + 4p_2 + \mathrm{i}p_1 & p_1 + \mathrm{i}p_2 & 0 \\ p_2 & 2 & p_2 \\ p_1 & 0 & 2 + p_1 \end{pmatrix} \tag{50}$$

and

$$B(\mathbf{p}) = \begin{pmatrix} 1 + p_1 + \mathrm{i}p_2 & p_1 & 0 \\ p_2 & 2 + p_1 & p_1 \\ 0 & 0 & 2 + p_2 \end{pmatrix}, \tag{51}$$

where $\mathbf{p} = (p_1, p_2)^T \in \mathbb{R}^2$ *and* $\mathrm{i} = \sqrt{-1}$.

We observe that

$$A(\mathbf{0}) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B(\mathbf{0}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

where $(\sigma_1, 1) = (3, 1)$ is a simple nonzero finite generalized singular value of $\{A(\mathbf{0}), B(\mathbf{0})\}$ and

$$\mathbf{u}_1 = \mathbf{v}_1 = \mathbf{q}_1 = (1, 0, 0)^T.$$

Using (50) and (51), we get

$$\left(\frac{\partial A(\mathbf{p})}{\partial p_1}\right)_{\mathbf{p}=\mathbf{0}} = \begin{pmatrix} 2+\mathrm{i} & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \left(\frac{\partial A(\mathbf{p})}{\partial p_2}\right)_{\mathbf{p}=\mathbf{0}} = \begin{pmatrix} 4 & \mathrm{i} & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\left(\frac{\partial B(\mathbf{p})}{\partial p_1}\right)_{\mathbf{p}=\mathbf{0}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \left(\frac{\partial B(\mathbf{p})}{\partial p_2}\right)_{\mathbf{p}=\mathbf{0}} = \begin{pmatrix} 1+\mathrm{i} & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By (21), (47) and (49),

$$s_{p_1}(3) = 2, \quad s_{p_2}(3) = 1, \quad s_p(3) = \sqrt{5}.$$

Next, we present an example to show how the second order perturbation expansions of simple nonzero finite generalized singular values work.

Example 4.3 *Let*

$$A(\mathbf{p}) = \begin{pmatrix} 6 & -2 \\ \frac{4}{p_1+\mathrm{i}p_2+2} & 6 \end{pmatrix} \quad \text{and} \quad B(\mathbf{p}) = \begin{pmatrix} 5 & \frac{3}{p_1+\mathrm{i}p_2+1} \\ 0 & 4 \end{pmatrix},$$

where $\mathbf{p} = (p_1, p_2)^T \in \mathbb{R}^2$.

Obviously, the matrix pair $\{A(\mathbf{0}), B(\mathbf{0})\}$ is given by

$$A(\mathbf{0}) = \begin{pmatrix} 6 & -2 \\ 2 & 6 \end{pmatrix} \quad \text{and} \quad B(\mathbf{0}) = \begin{pmatrix} 5 & 3 \\ 0 & 4 \end{pmatrix},$$

which has the following GSVD:

$$U^H A(\mathbf{0}) Q = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad V^H B(\mathbf{0}) Q = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

where

$$U = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}, \quad V = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \quad Q = \frac{1}{2\sqrt{5}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Thus one has

$$\sigma_1 = 2, \quad \Lambda_2 = \Sigma_2 = 2, \quad \Phi_1 = \Phi_2 = \frac{1}{6}, \quad \Phi_3 = \frac{1}{3}, \quad \Phi_4 = \frac{2}{3},$$

$$\mathbf{u}_1 = \frac{1}{\sqrt{5}}(2, -1)^T, \quad U_2 = \frac{1}{\sqrt{5}}(1, 2)^T, \quad \mathbf{v}_1 = \frac{1}{\sqrt{5}}(1, -2)^T,$$

$$V_2 = \frac{1}{\sqrt{5}}(2, 1)^T, \quad \mathbf{q}_1 = \frac{1}{2\sqrt{5}}(1, -1)^T, \quad Q_2 = \frac{1}{2\sqrt{5}}(1, 1)^T,$$

$$\left(\frac{\partial A(\mathbf{p})}{\partial p_1}\right)_{\mathbf{p}=\mathbf{0}} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \left(\frac{\partial A(\mathbf{p})}{\partial p_2}\right)_{\mathbf{p}=\mathbf{0}} = \begin{pmatrix} 0 & 0 \\ -\mathrm{i} & 0 \end{pmatrix},$$

$$\left(\frac{\partial B(\mathbf{p})}{\partial p_1}\right)_{\mathbf{p}=\mathbf{0}} = \begin{pmatrix} 0 & -3 \\ 0 & 0 \end{pmatrix}, \quad \left(\frac{\partial B(\mathbf{p})}{\partial p_2}\right)_{\mathbf{p}=\mathbf{0}} = \begin{pmatrix} 0 & -3\mathrm{i} \\ 0 & 0 \end{pmatrix},$$

$$\left(\frac{\partial^2 A(\mathbf{p})}{\partial p_1^2}\right)_{\mathbf{p}=\mathbf{0}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \left(\frac{\partial^2 B(\mathbf{p})}{\partial p_1^2}\right)_{\mathbf{p}=\mathbf{0}} = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix},$$

$$\left(\frac{\partial^2 A(\mathbf{p})}{\partial p_1 \partial p_2}\right)_{\mathbf{p}=\mathbf{0}} = \begin{pmatrix} 0 & 0 \\ \mathrm{i} & 0 \end{pmatrix}, \quad \left(\frac{\partial^2 B(\mathbf{p})}{\partial p_1 \partial p_2}\right)_{\mathbf{p}=\mathbf{0}} = \begin{pmatrix} 0 & 3\mathrm{i} \\ 0 & 0 \end{pmatrix},$$

$$\left(\frac{\partial^2 A(\mathbf{p})}{\partial p_2^2}\right)_{\mathbf{p}=\mathbf{0}} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \left(\frac{\partial^2 B(\mathbf{p})}{\partial p_2^2}\right)_{\mathbf{p}=\mathbf{0}} = \begin{pmatrix} 0 & -3 \\ 0 & 0 \end{pmatrix}.$$

Using (21) and (25), a simple calculation yields

$$\left(\frac{\partial \sigma_1(\mathbf{p})}{\partial p_1}\right)_{\mathbf{p}=\mathbf{0}} = -0.5, \quad \left(\frac{\partial \sigma_1(\mathbf{p})}{\partial p_2}\right)_{\mathbf{p}=\mathbf{0}} = 0,$$

and

$$\left(\frac{\partial^2 \sigma_1(\mathbf{p})}{\partial p_1^2}\right)_{\mathbf{p}=\mathbf{0}} = 0.7017, \quad \left(\frac{\partial^2 \sigma_1(\mathbf{p})}{\partial p_1 \partial p_2}\right)_{\mathbf{p}=\mathbf{0}} = 0, \quad \left(\frac{\partial^2 \sigma_1(\mathbf{p})}{\partial p_2^2}\right)_{\mathbf{p}=\mathbf{0}} = 0.44.$$

Therefore, $\sigma_1(\mathbf{p})$ has the expansion

$$\sigma_1(\mathbf{p}) = 2 - 0.5p_1 + 0.3508p_1^2 + 0.22p_2^2 + O(\|\mathbf{p}\|^3)$$

in a neighborhood of the origin, where $\|\cdot\|$ denotes the Euclidean vector norm.

These examples show that our results are effective for evaluating the sensitivity and the second order Taylor expansions of simple nonzero finite generalized singular values.

5 Future work

In this paper, we give the first order and second order partial derivatives of simple nonzero finite generalized singular values of a complex matrix pair analytically dependent on several parameters. These results may be used to investigate the effectiveness of the GSVD-based methods for practical applications. An interesting problem is to discuss the sensitivity analysis and the second order perturbation expansions of zero generalized singular values, infinite generalized singular values and multiple generalized singular values analytically dependent on several parameters. This needs further study.

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