A Regularized Directional Derivative-Based Newton Method for Inverse Singular Value Problems

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Abstract

In this paper, we give a regularized directional derivative-based Newton method for solving the inverse singular value problem. The proposed method is also globalized by employing the directional derivative-based Wolfe line search conditions. Under some mild assumptions, The global and quadratic convergence of our method is established. To improve the practical effectiveness, we also propose a hybrid method for solving the inverse singular value problem. We show that the hybrid method converges locally quadratically and globally in the sense that a stationary point of a merit function for the inverse singular value problem is computed. Numerical tests demonstrate that the proposed hybrid method is very effective for solving the inverse singular value problem with distinct and multiple singular values.

Keywords Inverse singular value problem, generalized Newton method, directional derivative, Wolfe conditions, global convergence

AMS subject classification. 65F18, 65F15, 15A18, 49J52, 90C56

1 Introduction

The inverse problem of reconstructing a matrix from the given singular values, i.e. the inverse singular value problem, has a growing importance in many applications such as the optimal sequence design for direct-spread code division multiple access [40], the passivity enforcement in nonlinear circuit simulation [34], the constructions of Toeplitz-related matrices from prescribed singular values[1, 2, 13], the inverse problem in some quadratic group [27], and the construction of nonnegative matrices, positive matrices and anti-bisymmetric matrices with prescribed singular values [23, 24, 43], and others [31].

In this paper, we consider the inverse singular value problem defined as follows.

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ISVP: *Let* $\{A_i\}_{i=0}^n$ *be* $n+1$ *real* $m \times n$ *matrices* $(m \geq n)$ *and let* $\sigma_1^* \geq \sigma_2^* \geq \cdots \geq \sigma_n^*$ *be* n *nonnegative numbers. Find a vector* $\mathbf{c}^* \in \mathbb{R}^n$ *such that the singular values of the matrix*

$$A(\mathbf{c}^*) := A_0 + \sum_{i=1}^{n} c_i^* A_i \tag{1}$$

are exactly $\sigma_1^*, \ldots, \sigma_n^*$.

This is a special kind of inverse singular value problems, which was originally proposed by Chu [5]. It is easy to check that the ISVP can be converted into the following inverse eigenvalue problem (IEP): Given $n+1$ real $m \times n$ matrices $\{A_i\}_{i=0}^n$ and n numbers $\sigma_1^* \geq \sigma_2^* \geq \cdots \geq \sigma_n^* \geq 0$, find a vector $\mathbf{c}^* \in \mathbb{R}^n$ such that the eigenvalues of the matrix

$$B(\mathbf{c}^*) := \begin{bmatrix} 0 & A_0 \\ A_0^T & 0 \end{bmatrix} + \sum_{i=1}^{n} c_i^* \begin{bmatrix} 0 & A_i \\ A_i^T & 0 \end{bmatrix}$$

are exactly $\sigma_1^*, \ldots, \sigma_n^*$, $-\sigma_1^*, \ldots, -\sigma_n^*$, and zero of multiplicity $m-n$. Therefore, the ISVP is solvable if and only if the above IEP is solvable. To our knowledge, there is no literature on the existence and uniqueness questions for the ISVP. For more discussion on the ISVP, one may refer to [6, 7].

However, there exist some numerical methods developed for computational purpose. For any $\mathbf{c} \in \mathbb{R}^n$, denote the singular values of $A(\mathbf{c})$ by $\{\sigma_i(A(\mathbf{c}))\}_{i=1}^n$ with the ordering $\sigma_1(A(\mathbf{c})) \geq \sigma_2(A(\mathbf{c})) \geq \cdots \geq \sigma_n(A(\mathbf{c})) \geq 0$. Then, the ISVP is to find a solution $\mathbf{c}^* \in \mathbb{R}^n$ of the following nonlinear equation:

$$\mathbf{f}(\mathbf{c}) := (\sigma_1(A(\mathbf{c})) - \sigma_1^*, \sigma_2(A(\mathbf{c})) - \sigma_2^*, \ldots, \sigma_n(A(\mathbf{c})) - \sigma_n^*)^T = \mathbf{0}. \tag{2}$$

Based on the nonlinear equation (2), Newton-type methods and an Ulm-like method were developed for solving the ISVP [4, 5, 41]. The advantage of Newton-type methods and Ulm's method lies in its superlinear convergence. However, these methods converge only locally.

Based on the generalized Jacobian and the directional derivative, many nonsmooth versions of Newton's method were developed for solving nonsmooth equations [26, 29, 30] and their globalized versions were proposed for solving nonlinear complementarity problems [9, 20, 21, 26]. A typical nonlinear complementarity problem is to find a vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\mathbf{x} \geq 0, \quad F(\mathbf{x}) \geq 0, \quad \mathbf{x}^T F(\mathbf{x}) = 0,$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable function and $\mathbf{x} \geq 0$ means that $x_j \geq 0$ for $j = 1, \ldots, n$. For various applications of nonlinear complementarity problems, the interested reader may refer to the survey papers by Harker and Pang [18] and Ferris and Pang [11].

The directional differentiability of all singular values of a rectangular matrix was discussed in [35] and the directional derivatives of all eigenvalues of a symmetric matrix were provided in [19]. Also, it was showed in [39] that all singular values of a rectangular matrix are strongly semismooth and the generalized Newton method in [38] can be employed to solve the ISVP. The regularization techniques were also used in generalized Newton methods for nonlinear complementary problems [10, 37]. Motivated by this, in this paper, we propose a regularized directional derivative-based Newton method for solving the ISVP. We first give the explicit formulas of the directional derivatives of sums of singular values of $A(\mathbf{c})$ for any $\mathbf{c} \in \mathbb{R}^n$. Then, we present a regularized directional derivative-based Newton method for the ISVP. Our method is globalized by the directional derivative-based Wolfe line search conditions. Under the assumptions that all elements in the generalized Jacobian of a nonlinear function of the ISVP are nonsingular at a solution of the ISVP (see Assumption 4.3 below) and the subset $\mathcal{D} := \{\mathbf{c} \in \mathbb{R}^n : A(\mathbf{c}) \text{ has distinct singular values}\}$ of $\mathbb{R}^n$ is dense in $\mathbb{R}^n$, the global and quadratical convergence of our method is established. To further improve the feasibility, sparked by [17, 29], we also propose a hybrid method for solving the ISVP by combining the directional derivative-based generalized Newton method with an Armijo-like line search based on a merit function of the ISVP. Under the same assumptions, the hybrid method is shown to converge quadratically and globally (in the sense of finding a stationary point of the merit function for the ISVP). Compared to the generalized Newton method in [38], our method makes use of the directional derivatives of singular values of $A(\mathbf{c})$ instead of their generalized Jacobians [8] and the direction generated by the directional derivative-based generalized Newton equation can be applied to globalizing our method by using the Armijo-like line search or the Wolfe line search. Some numerical tests are reported to illustrate the effectiveness of the proposed hybrid method for solving the ISVP with both distinct and multiple singular values.

This paper is organized as follows. In Section 2 we review some preliminary results on nonsmooth analysis. In Section 3 we propose a regularized directional derivative-based Newton method for solving the ISVP. In Section 4 we give the convergence analysis. In Section 5 we present a hybrid method for solving the ISVP. In Section 6 we report some numerical experiments.

2 Preliminaries

In the section, we review some necessary concepts and basic properties related to Lipschitz continuous functions. Let $\mathcal{X}$ be a finite dimensional real vector space, equipped with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. We note that a function $\Phi : \mathcal{X} \to \mathbb{R}^n$ is directionally differentiable at $x \in \mathcal{X}$ if the directional derivative

$$\Phi'(x; \Delta x) = \lim_{t \downarrow 0} \frac{\Phi(x + t\Delta x) - \Phi(x)}{t}$$

exists for all $\Delta x \in \mathcal{X}$. In the following, we recall the definition of B-differentiability [32].

Definition 2.1 *A function $\Phi : \mathcal{X} \to \mathbb{R}^n$ is said to be B-differentiable at $x \in \mathcal{X}$ if it is directionally differentiable at x and*

$$\lim_{\Delta x \to 0} \frac{\Phi(x + \Delta x) - \Phi(x) - \Phi'(x; \Delta x)}{\|\Delta x\|} = 0.$$

For a locally Lipschitzian function $\Phi : \mathcal{X} \to \mathbb{R}^n$, it was shown that the B-differentiability of Φ is equivalent to its directional differentiability [36].

Let $\Phi : \mathcal{X} \to \mathbb{R}^n$ be a locally Lipschitz continuous function. We study the following nonlinear equation:

$$\Phi(\mathbf{x}) = \mathbf{0}. \tag{3}$$

If Φ is nonsmooth, one may use Clarke's generalized Jacobian-based Newton method for solving (3) [30]. Notice that Φ is Fréchet differentiable almost everywhere [33]. Clarke's generalized Jacobian $\partial\Phi(x)$ [8] of Φ at $x \in \mathcal{X}$ is defined by:

$$\partial\Phi(x) := \operatorname{co} \partial_B \Phi(x) := \operatorname{co}\{\lim \Phi'(x^j) \,|\, x^j \to x,\ \Phi \text{ is differentiable at } x^j\}.$$

where "co" means the convex hull and $\Phi'(x)$ means the Jacobian of Φ at $x \in \mathcal{X}$.

Then, the generalized Newton method for solving (3) is given as follows [30]: Set $x^{k+1} = x^k + d^k$, where

$$d^k = -V_k^{-1}\Phi(x^k), \quad V_k \in \partial\Phi(x^k). \tag{4}$$

If Φ is both locally Lipschitz continuous and directionally differentiable, then a directional derivative-based Newton method for solving (3) is given as follows [26, 29]: Set $x^{k+1} = x^k + d^k$, where d^k solves

$$\Phi(x^k) + \Phi'(x^k; d^k) = 0. \tag{5}$$

We point out that (5) coincides with (4) since there is a relation between Clarke's generalized Jacobian and the directional derivative of a locally Lipschitzian function [30, Lemma 2.2].

Lemma 2.2 *Let $\Phi : \mathcal{X} \to \mathbb{R}^m$ be a locally Lipschitzian function and directionally differentiable on a neighborhood of $x \in \mathcal{X}$. Then $\Phi'(x; \cdot)$ is Lipschitzian and for any $\Delta x \in \mathcal{X}$, there exists a $V \in \partial\Phi(x)$ such that $\Phi'(x; \Delta x) = V\Delta x$.*

To give the convergence analysis of (4) or (5), we need the following definition of semismoothness. For the original concept of semismoothness for functionals and vector-valued functions, one may refer to [22, 30].

Definition 2.3 *Let $\Phi : \mathcal{X} \to \mathbb{R}^m$ be a locally Lipschitz continuous function.*

(a) *Φ is said to be semismooth at $x \in \mathcal{X}$ if for any $V \in \partial\Phi(x + h)$ and $h \to 0$,*

$$V(h) - \Phi'(x; h) = o(\|h\|).$$

(b) *Φ is said to be strongly semismooth at x if for any $V \in \partial\Phi(x + h)$ and $h \to 0$,*

$$V(h) - \Phi'(x; h) = O(\|h\|^2).$$

Definition 2.3 shows that a (strongly) semismooth function must be B-differentiable. Therefore, if Φ is semismooth (strongly semismooth, respectively) at $x \in \mathcal{X}$, then for any $h \to 0$,

$$\Phi(x + h) - \Phi(x) - \Phi'(x; h) = o(\|h\|)\ (O(\|h\|^2), \text{ respectively}).$$

We now give the convergence result of (4) [30, Theorem 3.2]. The result on superlinear convergence of (5) can be found in [29, Theorem 4.3].

Proposition 2.4 *Let $\Phi : \mathcal{X} \to \mathbb{R}^n$ be a locally Lipschitz continuous function. Let $\bar{x} \in \mathcal{X}$ be an accumulation point of a sequence $\{x^k\}$ generated by* (4) *and $\Phi(\bar{x}) = 0$. Assume that Φ is semismooth at $\bar{x}$ and all $V \in \partial F(\bar{x})$ are nonsingular. Then the sequence $\{x^k\}$ converges to $\bar{x}$ superlinearly provided that the initial guess x^0 is sufficiently close to $\bar{x}$. Moreover, if Φ is strongly semismooth at $\bar{x}$, then the convergence rate is quadratic.*

On the nonsingularity of Clarke's generalized Jacobian of a locally Lipschitz continuous function, we have the following lemma [29, 30].

Lemma 2.5 *Let $\Phi : \mathcal{X} \to \mathbb{R}^m$ be a locally Lipschitz continuous function. Suppose that all elements in $\partial_B \Phi(x)$ are nonsingular. Then there exist a constant κ and a neighborhood $\mathcal{N}(x)$ of x such that, for any $y \in \mathcal{N}(x)$ and any $V \in \partial_B \Phi(y)$, V is nonsingular and $\|V^{-1}\| \leq \kappa$. If Φ is semismooth at y, then, for any $h \in \mathcal{X}$, there exists $V \in \partial_B \Phi(y)$ such that $\Phi'(y; h) = Vh$ and $\|h\| \leq \kappa \|\Phi'(y; h)\|$.*

Finally, we have the following lemma on the (strongly semismoothness) semismoothness of composite functions [12, 22].

Lemma 2.6 *Suppose that $\Phi : \mathcal{X} \to \mathbb{R}^m$ is* (*strongly semismooth*) *semismooth at $x \in \mathcal{X}$ and $\Psi : \mathbb{R}^m \to \mathbb{R}^q$ is* (*strongly semismooth*) *semismooth at $\Phi(x)$. Then the composite function $\Upsilon = \Psi \circ \Phi$ is* (*strongly semismooth*) *semismooth at x.*

3 A regularized directional derivative-based Newton method

In this section, we first reformulate the ISVP as a new nonsmooth equation and then propose a regularized directional derivative-based Newton method for solving the nonsmooth equation. In what follows, let $\mathcal{M}^{m\times n}$, $\mathcal{S}^n$ and $\mathcal{O}(n)$ denote the set of all m-by-n real matrices, the set of all n-by-n real symmetric matrices and the set of all n-by-n orthogonal matrices, respectively. Denote by M^T the transpose of a matrix M. Let I_n be the identity matrix of order n. Let $\{\sigma_j(M)\}_{j=1}^n$ stand for the singular values of a matrix $M \in \mathcal{M}^{m\times n}$ with $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \sigma_n(M) \geq 0$. In this case, we call $\sigma_j(M)$ the j-th largest singular value of M.

Define the linear operator $\Theta : \mathcal{M}^{m\times n} \to \mathcal{S}^{m+n}$ by

$$\Theta(Z) := \begin{bmatrix} 0 & M \\ M^T & 0 \end{bmatrix} \quad \forall M \in \mathcal{M}^{m\times n}.$$

The singular value decomposition of a matrix $M \in \mathcal{M}^{m\times n}$ $(m \geq n)$ is given by [16]

$$P^T M Q = \begin{bmatrix} \Sigma(M) \\ 0 \end{bmatrix}, \quad \Sigma(M) = \mathrm{diag}(\sigma_1(M), \sigma_2(M), \ldots, \sigma_n(M)),$$

where $P \in \mathcal{O}(m)$, $Q \in \mathcal{O}(n)$, and $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_n(M) \geq 0$ are the singular values of M. Partition P by $P = [P^{(1)}, P^{(2)}]$, where $P^{(1)} \in \mathcal{M}^{m\times n}$ and $P^{(2)} \in \mathcal{M}^{m\times(m-n)}$. Let $U \in \mathcal{O}(m+n)$ be defined by

$$U := \frac{1}{\sqrt{2}} \begin{bmatrix} P^{(1)} & -P^{(1)} & \sqrt{2}P^{(2)} \\ Q & Q & 0 \end{bmatrix}. \tag{6}$$

It is easy to check that the matrix $\Theta(M)$ admits the following spectral decomposition:

$$\Theta(M) = U \begin{bmatrix} \Sigma(M) & 0 & 0 \\ 0 & -\Sigma(M) & 0 \\ 0 & 0 & 0 \end{bmatrix} U^T. \tag{7}$$

That is, $\Theta(M)$ has the eigenvalues $\pm\sigma_i(M), i = 1, 2, \ldots, n$, and 0 of multiplicity $m - n$.

For $j = 1, \ldots, n$, define

$$\varphi_j(M) := \sum_{i=1}^{j} \sigma_i(M) \quad \forall M \in \mathcal{M}^{m\times n}.$$

By using the strong semismoothness of all eigenvalues of a real symmetric matrix [38], the linearity of $\Theta(\cdot)$ and Lemma 2.6, we have the following result on the strong semismoothness of $\{\sigma_j(\cdot)\}_{j=1}^n$ and $\{\varphi_j(\cdot)\}_{j=1}^n$ [39].

Lemma 3.1 *The functions $\{\sigma_j(\cdot)\}_{j=1}^n$ and $\{\varphi_j(\cdot)\}_{j=1}^n$ are strongly semismooth functions over $\mathcal{M}^{m\times n}$.*

Now, we consider the ISVP. For the convenience of theoretical analysis, instead of (2), in what follows, we focus on a new nonsmooth reformulation of the ISVP: Find a solution $\mathbf{c}^* \in \mathbb{R}^n$ of the following nonsmooth equation:

$$\mathbf{g}(\mathbf{c}) := \boldsymbol{\varphi}(A(\mathbf{c})) - \boldsymbol{\varphi}^* = (\varphi_1(A(\mathbf{c})) - \varphi_1^*, \varphi_2(A(\mathbf{c})) - \varphi_2^*, \ldots, \varphi_n(A(\mathbf{c})) - \varphi_n^*)^T = \mathbf{0}, \tag{8}$$

where

$$\varphi_j(A(\mathbf{c})) = \sum_{i=1}^{j} \sigma_i(A(\mathbf{c})) \quad \forall \mathbf{c} \in \mathbb{R}^n \quad \text{and} \quad \varphi_j^* = \sum_{i=1}^{j} \sigma_i^*, \quad j = 1, \ldots, n.$$

We point out that the function $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^n$ defined in (8) is a composite nonsmooth function, where $\boldsymbol{\varphi} : \mathcal{M}^{m\times n} \to \mathbb{R}^n$ is strongly semismooth and the linear function $A : \mathbb{R}^n \to \mathcal{M}^{m\times n}$ defined in (1) is continuously differentiable. It follows from Lemma 2.6 that $\mathbf{g}$ is strongly semismooth and thus for any $\mathbf{c}, \mathbf{h} \in \mathbb{R}^n$,

$$\mathbf{g}'(\mathbf{c}; \mathbf{h}) = \boldsymbol{\varphi}(A(\mathbf{c}); A'(\mathbf{c})\mathbf{h}), \quad A'(\mathbf{c})\mathbf{h} = \sum_{j=1}^{n} h_j \frac{\partial A(\mathbf{c})}{\partial c_j}.$$

For any $\mathbf{c} \in \mathbb{R}^n$, define the sets $\mathcal{P}(\mathbf{c})$ and $\mathcal{Q}(\mathbf{c})$ by

$$(\mathcal{P}(\mathbf{c}), \mathcal{Q}(\mathbf{c})) := \left\{ (P(\mathbf{c}), Q(\mathbf{c})) \in \mathcal{O}(m) \times \mathcal{O}(n) : P(\mathbf{c})^T A(\mathbf{c}) Q(\mathbf{c}) = \begin{bmatrix} \Sigma(A(\mathbf{c})) \\ 0 \end{bmatrix} \right\}.$$

where $\Sigma(A(\mathbf{c})) := \operatorname{diag}(\sigma_1(A(\mathbf{c})), \sigma_2(A(\mathbf{c})), \ldots, \sigma_n(A(\mathbf{c})))$. For any $\mathbf{c} \in \mathbb{R}^n$, let

$$P(\mathbf{c}) := [\mathbf{p}_1(\mathbf{c}), \ldots, \mathbf{p}_m(\mathbf{c})] \in \mathcal{P}(\mathbf{c}) \quad \text{and} \quad Q(\mathbf{c}) := [\mathbf{q}_1(\mathbf{c}), \ldots, \mathbf{q}_n(\mathbf{c})] \in \mathcal{Q}(\mathbf{c}).$$

For the j-th largest singular value $\sigma_j(A(\mathbf{c}))$ of $A(\mathbf{c})$ with multiplicity r_j, let s_j be the number of singular values, ranking before j, which are equal to $\sigma_j(A(\mathbf{c}))$ and define

$$P_{j1}(\mathbf{c}) := [\mathbf{p}_1(\mathbf{c}), \ldots, \mathbf{p}_{j-s_j}(\mathbf{c})], \quad P_{j2}(\mathbf{c}) := [\mathbf{p}_{j-s_j+1}(\mathbf{c}), \ldots, \mathbf{p}_{j+r_j-s_j}(\mathbf{c})],$$

$$Q_{j1}(\mathbf{c}) := [\mathbf{q}_1(\mathbf{c}), \ldots, \mathbf{q}_{j-s_j}(\mathbf{c})], \quad Q_{j2}(\mathbf{c}) := [\mathbf{q}_{j-s_j+1}(\mathbf{c}), \ldots, \mathbf{q}_{j+r_j-s_j}(\mathbf{c})].$$

By using (6), (7), and [19, Theorem 4.4], we can easily derive the following proposition on the directional derivative of $\mathbf{g}$ in (8).

Proposition 3.2 *For any $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{h} \in \mathbb{R}^n$, we set*

$$E_1'(\mathbf{h}) := P_{j1}^T(\mathbf{c})\Big(\sum_{i=1}^n h_i \partial A(\mathbf{c})/\partial c_i\Big) Q_{j1}(\mathbf{c}) \quad and \quad E_2'(\mathbf{h}) := P_{j2}^T(\mathbf{c})\Big(\sum_{i=1}^n h_i \partial A(\mathbf{c})/\partial c_i\Big) Q_{j2}(\mathbf{c}).$$

Then, the directional derivative of $\mathbf{g}(\cdot) := (g_1(\cdot), \ldots, g_n(\cdot))$ at $\mathbf{c} \in \mathbb{R}^n$ in the direction $\mathbf{h} \in \mathbb{R}^n$ is given by

$$g_j'(\mathbf{c}; \mathbf{h}) = \operatorname{tr}\left(\frac{1}{2}\Big(E_1'(\mathbf{h}) + E_1'(\mathbf{h})^T\Big)\right) + \sum_{i=1}^{s_j} \mu_i \left(\frac{1}{2}\Big(E_2'(\mathbf{h}) + E_2'(\mathbf{h})^T\Big)\right)$$

for $j = 1, \ldots, n$, where "$\operatorname{tr}(\cdot)$" and $\mu_i(\cdot)$ denote the trace and the i-th largest eigenvalue of a matrix, respectively.

Proposition 3.2 shows that for any $\mathbf{c} \in \mathbb{R}^n$, the directional derivative $\mathbf{g}'(\mathbf{c}; \mathbf{h})$ in the direction $\mathbf{h} \in \mathbb{R}^n$ can be formulated easily once the singular value decomposition of $A(\mathbf{c})$ is computed. This motivates us to propose the following directional derivative-based Newton method for solving the ISVP: Set $\mathbf{c}^{k+1} = \mathbf{c}^k + \mathbf{d}^k$, where $\mathbf{d}^k$ solves

$$\mathbf{g}(\mathbf{c}^k) + \mathbf{g}'(\mathbf{c}^k; \mathbf{d}^k) = \mathbf{0}. \tag{9}$$

Notice that $\mathbf{g}$ is strongly semismooth. By Lemma 2.5, there exists a $V_k \in \partial_B \mathbf{g}(\mathbf{c}^k)$ such that $\mathbf{g}'(\mathbf{c}^k; \mathbf{d}^k) = V_k \mathbf{d}^k$. Hence, the direction derivative-based Newton method (9) coincides with the generalized Newton method [29, 30]

$$\mathbf{c}^{k+1} = \mathbf{c}^k - V_k^{-1} \mathbf{g}(\mathbf{x}^k), \quad V_k \in \partial_B \mathbf{g}(\mathbf{c}^k).$$

Therefore, the equation (9) can be solved by an iterative method (e.g., the transpose-free quasi-minimal residual (TFQMR) method [14]). In addition, by Proposition 2.4, we know that (9) converges locally quadratically if the initial guess $\mathbf{c}^0$ is sufficiently close to a zero $\bar{\mathbf{c}}$ of $\mathbf{g}$ under the nonsingularity assumption of $\partial \mathbf{g}(\bar{\mathbf{c}})$.

However, when the initial guess $\mathbf{c}^0$ is far away from a zero $\bar{\mathbf{c}}$ of $\mathbf{g}$, the directional derivative-based Newton equation (9) is not necessarily solvable even though we assume that all elements in $\partial \mathbf{g}(\bar{\mathbf{c}})$ are nonsingular. To improve the global solvability and practical effectiveness of the directional derivative-based Newton equation (9), one may use the similar regularization techniques for nonlinear complementary problems [10, 37], i.e., we replace $\mathbf{g}$ by $\mathbf{g}_\epsilon$, where

$$\mathbf{g}_\epsilon(\mathbf{c}) := \mathbf{g}(\mathbf{c}) + \epsilon \mathbf{c} \quad \forall (\epsilon, \mathbf{c}) \in \mathbb{R} \times \mathbb{R}^n.$$

For the ISVP, we define a regularized function $\mathbf{w} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$\mathbf{w}(\mathbf{z}) := \begin{bmatrix} \epsilon \\ \mathbf{g}_\epsilon(\mathbf{c}) \end{bmatrix}, \quad \mathbf{z} := (\epsilon, \mathbf{c}) \in \mathbb{R} \times \mathbb{R}^n.$$

Then, $\mathbf{w}$ is strongly semismooth and $\mathbf{c}^*$ solves (8) if and only if $(0, \mathbf{c}^*)$ solves $\mathbf{w}(\mathbf{z}) = \mathbf{0}$. To develop global convergence, we also define the merit function $\omega : \mathbb{R}^{n+1} \to \mathbb{R}$ by

$$\omega(\mathbf{z}) := \frac{1}{2}\|\mathbf{w}(\mathbf{z})\|^2, \quad \mathbf{z} = (\epsilon, \mathbf{c}) \in \mathbb{R} \times \mathbb{R}^n. \tag{10}$$

By Lemma 2.6 and Lemma 3.1, it is easy to know that ω is strongly semismooth. By the chain rule, the directional derivative of ω at any $\mathbf{z} \in \mathbb{R}^{n+1}$ is given by

$$\omega'(\mathbf{z}; \mathbf{d}) = \mathbf{w}(\mathbf{z})^T \mathbf{w}'(\mathbf{z}; \mathbf{d}), \quad \mathbf{w}'(\mathbf{z}; \mathbf{d}) = \begin{bmatrix} \Delta\epsilon \\ \mathbf{g}'(\mathbf{c}; \Delta\mathbf{c}) + \epsilon\Delta\mathbf{c} + \Delta\epsilon\mathbf{c} \end{bmatrix} \tag{11}$$

for all $\mathbf{d} := (\Delta\epsilon, \Delta\mathbf{c}) \in \mathbb{R}^{n+1}$.

By Proposition 3.2 and (11), we know that for any $\mathbf{z} = (\epsilon, \mathbf{c}) \in \mathbb{R} \times \mathbb{R}^n$, the directional derivative $\mathbf{w}'(\mathbf{z}; \mathbf{d})$ in the direction $\mathbf{d} := (\Delta\epsilon, \Delta\mathbf{c}) \in \mathbb{R}^{n+1}$ can be formulated explicitly since $\mathbf{w}'(\mathbf{c}; \Delta\mathbf{c})$ is available once the singular value decomposition of $A(\mathbf{c})$ is obtained. Also, sparked by the generalized Newton method in [26] for solving B-differentiable equations, we propose the following regularized directional derivative-based Newton method for solving $\mathbf{w}(\mathbf{z}) = \mathbf{0}$.

Algorithm I: A Regularized Directional Derivative-Based Newton Method

Step 0. *Given* $0 < \lambda_1 < \lambda_2 < 1$, $\eta \in (0, (1-\lambda_2)/(1+\lambda_2))$ *and* $0 \neq \bar{\epsilon} \in (-1, 1)$. *Let* $\mathbf{z}^0 := (\epsilon^0, \mathbf{c}^0)$ *be the initial point, where* $\epsilon^0 := \bar{\epsilon}$ *and* $\mathbf{c}^0 \in \mathbb{R}^n$ *is arbitrary.* $k := 0$.

Step 1. *If* $\|\mathbf{w}(\mathbf{z}^k)\| = 0$, *then stop. Otherwise, go to Step* 2.

Step 2. *Apply an iterative method (e.g., the TFQMR method) to solve*

$$\mathbf{w}(\mathbf{z}^k) + \mathbf{w}'(\mathbf{z}^k; \mathbf{d}^k) = \mathbf{0}. \tag{12}$$

for $\mathbf{d}^k := (\Delta\epsilon^k, \Delta\mathbf{c}^k) \in \mathbb{R}^{n+1}$ *such that*

$$\|\mathbf{w}(\mathbf{z}^k) + \mathbf{w}'(\mathbf{z}^k; \mathbf{d}^k)\| \leq \eta_k \|\mathbf{w}(\mathbf{z}^k)\|, \tag{13}$$

and

$$\omega'(\mathbf{z}^k; \mathbf{d}^k) \leq -\eta_k \langle \mathbf{d}^k, \mathbf{d}^k \rangle, \tag{14}$$

where $\eta_k := \min\{\eta, \|\mathbf{w}(\mathbf{z}^k)\|\}$.

Step 3. *Compute the steplength* $\alpha_k > 0$ *satisfying the directional derivative-based standard Wolfe conditions* [25]*:*

$$\omega(\mathbf{z}^k + \alpha_k \mathbf{d}^k) \leq \omega(\mathbf{z}^k) + \lambda_1 \alpha_k \omega'(\mathbf{z}^k; \mathbf{d}^k), \tag{15}$$

$$\omega'(\mathbf{z}^k + \alpha_k \mathbf{d}^k; \mathbf{d}^k) > \lambda_2 \omega'(\mathbf{z}^k; \mathbf{d}^k). \tag{16}$$

Set $\mathbf{z}^{k+1} := \mathbf{z}^k + \alpha_k \mathbf{d}^k$.

Step 4. *Replace k by $k+1$ and go to Step 1.*

We point out that in Algorithm I, we use a line search that satisfies the directional derivative-based Wolfe conditions (15) and (16) instead of the directional derivative-based Armijo rule (15) since the directional derivative-based Armijo rule may not reduce the slope sufficiently [25].

4 Convergence analysis

In this section, we shall establish the global and quadratic convergence of Algorithm I. We first recall some necessary perturbation results, which can be found in [3, 4, 41].

Lemma 4.1 (see [3, Lemma 2]) *For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have*

$$\|A(\mathbf{x}) - A(\mathbf{y})\| \le \zeta \|\mathbf{x} - \mathbf{y}\|,$$

where $\zeta = (\sum\limits_{i=1}^{n} \|A_i\|^2)^{1/2}$.

Lemma 4.2 (see [3, Lemma 4]) *For any $\mathbf{x} \in \mathbb{R}^n$, let $\{\sigma_i(A(\mathbf{x}))\}_{i=1}^n$, $\{\mathbf{p}_i(\mathbf{x})\}_{i=1}^n$ and $\{\mathbf{q}_i(\mathbf{x})\}_{i=1}^n$ be the singular values and associated normalized left singular vectors and normalized right singular vectors of $A(\mathbf{x})$ respectively. Suppose that $\{\sigma_i(A(\mathbf{x}))\}_{i=1}^n$ are all distinct. Then there exist positive numbers δ_0 and ξ such that, for any $\mathbf{y} \in \mathbb{R}^n$ with $\|\mathbf{y} - \mathbf{x}\| \le \delta_0$,*

$$\|\mathbf{p}_i(\mathbf{y}) - \mathbf{p}_i(\mathbf{x})\| \le \xi \|\mathbf{y} - \mathbf{x}\| \quad \text{and} \quad \|\mathbf{q}_i(\mathbf{y}) - \mathbf{q}_i(\mathbf{x})\| \le \xi \|\mathbf{y} - \mathbf{x}\|, \quad 1 \le i \le n.$$

Notice that $\mathbf{w}$ is already strongly semismooth. By Proposition 2.4, to show the superlinear convergence of Algorithm I, we also need the following nonsingularity assumption on the generalized Jacobian $\partial_B \mathbf{w}(\cdot)$.

Assumption 4.3 *All elements in $\partial_B \mathbf{w}(\bar{\mathbf{z}})$ are nonsingular, where $\bar{\mathbf{z}}$ is an accumulation point of the sequence $\{\mathbf{z}^k\}$ generated by Algorithm I.*

Remark 4.4 *Let $\bar{\mathbf{z}} = (0, \bar{\mathbf{c}})$ is an accumulation point of the sequence $\{\mathbf{z}^k := (\epsilon^k, \mathbf{c}^k)\}$ generated by Algorithm I such that $\mathbf{w}(\bar{\mathbf{z}}) = \mathbf{0}$. Obviously,*

$$\partial_B \mathbf{w}(\bar{\mathbf{z}}) = \left\{ \begin{bmatrix} 1 & 0 \\ \bar{\mathbf{c}} & V \end{bmatrix} : V \in \partial_B \mathbf{g}(\bar{\mathbf{c}}) \right\}.$$

If $A(\bar{\mathbf{c}})$ has distinct singular values, then

$$\partial \mathbf{w}(\bar{\mathbf{z}}) = \partial_B \mathbf{w}(\bar{\mathbf{z}}) = \left\{ \begin{bmatrix} 1 & 0 \\ \bar{\mathbf{c}} & \mathbf{g}'(\bar{\mathbf{c}}) \end{bmatrix} \right\},$$

where for any $P(\bar{\mathbf{c}}) := [\mathbf{p}_1(\bar{\mathbf{c}}), \ldots, \mathbf{p}_m(\bar{\mathbf{c}})] \in \mathcal{P}(\bar{\mathbf{c}})$ and $Q(\bar{\mathbf{c}}) := [\mathbf{q}_1(\bar{\mathbf{c}}), \ldots, \mathbf{q}_n(\bar{\mathbf{c}})] \in \mathcal{Q}(\bar{\mathbf{c}})$, the Jacobian of $\mathbf{g}$ at $\bar{\mathbf{c}}$ is given by ([42, Theorem 1.9.3] *and* [41])

$$[\mathbf{g}'(\bar{\mathbf{c}})]_{j,l} := \sum_{i=1}^{j} (\mathbf{p}_i(\bar{\mathbf{c}}))^T A_l \mathbf{q}_i(\bar{\mathbf{c}}), \quad 1 \le j, l \le n. \tag{17}$$

In this case, Assumption 4.3 holds if and only if $\mathbf{g}'(\bar{\mathbf{c}})$ *is nonsingular.*

If for all k *sufficiently large,* $A(\mathbf{c}^k)$ *has distinct eigenvalues, then*

$$\partial \mathbf{w}(\mathbf{z}^k) = \partial_B \mathbf{w}(\mathbf{z}^k) = \left\{ \left[\begin{array}{cc} 1 & 0 \\ \mathbf{c}^k & \mathbf{g}'(\mathbf{c}^k) + \epsilon^k I_n \end{array} \right] \right\}.$$

In this case, Assumption 4.3 can be measured by

$$\limsup_{k\to\infty} \|\big(\mathbf{g}'(\mathbf{c}^k) + \epsilon^k I_n\big)^{-1}\| < \infty.$$

Therefore, in practice, Assumption 4.3 can be tested by checking whether the matrix $\mathbf{g}'(\mathbf{c}^k)+\epsilon^k I_n$ *is nonsingular for all* k *sufficiently large.*

Define

$$\mathcal{D} := \{\mathbf{c} \in \mathbb{R}^n : A(\mathbf{c}) \text{ has distinct singular values}\}.$$

To establish the global and quadratic convergence of Algorithm I, we prove the following preliminary lemma.

Lemma 4.5 *Let* $\bar{\mathbf{z}} \in \mathbb{R}^{n+1}$ *be an accumulation point of the sequence* $\{\mathbf{z}^k\}$ *generated by Algorithm I. Suppose that Assumption 4.3 holds and* $\mathcal{D}$ *is dense in* $\mathbb{R}^n$*. If equation (12) is solvable for* $\mathbf{d}^k \in \mathbb{R}^{n+1}$ *such that the conditions (13) and (14) are satisfied, then there exists a constant* $\kappa > 0$ *such that for all* k *sufficiently large,*

$$\|\mathbf{d}^k\| \le \kappa(1+\eta)\|\mathbf{w}(\mathbf{z}^k)\|, \tag{18}$$

$$\|\omega'(\bar{\mathbf{z}};\mathbf{d}^k) - \omega'(\mathbf{z}^k+\mathbf{d}^k;\mathbf{d}^k)\| \le L_1(\mathbf{z}^k,\mathbf{d}^k)\|\mathbf{z}^k+\mathbf{d}^k-\bar{\mathbf{z}}\|\|\mathbf{d}^k\|, \tag{19}$$

where $L_1(\mathbf{z}^k,\mathbf{d}^k) := L^2 + 2(2+\xi\zeta n\sqrt{n})\|\mathbf{w}(\mathbf{z}^k+\mathbf{d}^k)\|$ *and* L *is the Lipschitz constant of* $\mathbf{w}$ *around* $\bar{\mathbf{z}}$*.*

In addition, if $\mathbf{w}(\bar{\mathbf{z}}) = \mathbf{0}$*, then for any* $0 \le \delta \le 1/2$ *and for all* k *sufficiently large,*

$$\|\mathbf{z}^k+\mathbf{d}^k-\bar{\mathbf{z}}\| = O(\|\mathbf{z}^k-\bar{\mathbf{z}}\|^2), \tag{20}$$

$$\|\mathbf{w}(\mathbf{z}^k+\mathbf{d}^k)\| \le \delta\|\mathbf{w}(\mathbf{z}^k)\|, \tag{21}$$

Proof: We show the conclusions (18)–(21) as follows. We first prove (18). By the semismoothness of $\mathbf{w}$, Assumption 4.3, Lemma 2.5, we have, for all k sufficiently large, there exist a constant $\kappa > 0$ and a $V_k \in \partial_B \mathbf{w}(\mathbf{z}^k)$ such that

$$\mathbf{w}'(\mathbf{z}^k;\mathbf{z}^k-\bar{\mathbf{z}}) = V_k(\mathbf{z}^k-\bar{\mathbf{z}}), \quad \mathbf{w}'(\mathbf{z}^k;\mathbf{d}^k) = V_k\mathbf{d}^k, \quad \|V_k^{-1}\| \le \kappa. \tag{22}$$

By the hypothesis and (22), for all k sufficiently large,

$$\begin{aligned} \|\mathbf{d}^k\| &\le \|V_k^{-1}\|\|V_k\mathbf{d}^k + \mathbf{w}(\mathbf{z}^k) - \mathbf{w}(\mathbf{z}^k)\| \\ &\le \kappa\big(\|\mathbf{w}'(\mathbf{z}^k;\mathbf{d}^k) + \mathbf{w}(\mathbf{z}^k)\| + \|\mathbf{w}(\mathbf{z}^k)\|\big) \\ &\le \kappa(1+\eta_k)\|\mathbf{w}(\mathbf{z}^k)\| \le \kappa(1+\eta)\|\mathbf{w}(\mathbf{z}^k)\|, \end{aligned}$$

which confirms (18).

Next, we show (19). Notice that $\mathbf{d}^k = (\Delta\epsilon^k, \Delta\mathbf{c}^k) \in \mathbb{R}\times\mathbb{R}^n$ for all k. By the semismoothness of $\mathbf{w}$ and Lemma 2.5, we have for all k sufficiently large, there exist $V(\bar{\mathbf{c}}) \in \partial_B\mathbf{g}(\bar{\mathbf{c}})$ and $V(\mathbf{c}^k + \Delta\mathbf{c}^k) \in \partial_B\mathbf{g}(\mathbf{c}^k + \Delta\mathbf{c}^k)$ such that

$$\mathbf{g}'(\bar{\mathbf{c}}; \Delta\mathbf{c}^k) = V(\bar{\mathbf{c}})\Delta\mathbf{c}^k \quad \text{and} \quad \mathbf{g}'(\mathbf{c}^k + \Delta\mathbf{c}^k; \Delta\mathbf{c}^k) = V(\mathbf{c}^k + \Delta\mathbf{c}^k)\Delta\mathbf{c}^k.$$

By the definition of $\partial_B\mathbf{g}(\bar{\mathbf{c}})$ and $\partial_B\mathbf{g}(\mathbf{c}^k + \Delta\mathbf{c}^k)$, there exist $\mathbf{x}^k$, $\mathbf{y}^k \in \mathbb{R}^n$, at which $\mathbf{g}$ is differentiable (i.e., $A(\cdot)$ has distinct singular values at the points $\mathbf{x}^k$, $\mathbf{y}^k$), such that for all k sufficiently large,

$$\|\mathbf{x}^k - \bar{\mathbf{c}}\| \le \|\mathbf{c}^k + \Delta\mathbf{c}^k - \bar{\mathbf{c}}\|^2, \quad \|\mathbf{y}^k - (\mathbf{c}^k + \Delta\mathbf{c}^k)\| \le \|\mathbf{c}^k + \Delta\mathbf{c}^k - \bar{\mathbf{c}}\|^2, \tag{23}$$

and

$$\|V(\bar{\mathbf{c}}) - \mathbf{g}'(\mathbf{x}^k)\| \le \|\mathbf{c}^k + \Delta\mathbf{c}^k - \bar{\mathbf{c}}\|, \quad \|V(\mathbf{c}^k + \Delta\mathbf{c}^k) - \mathbf{g}'(\mathbf{y}^k)\| \le \|\mathbf{c}^k + \Delta\mathbf{c}^k - \bar{\mathbf{c}}\|. \tag{24}$$

By using (23), we get for all k sufficiently large,

$$\begin{aligned} \|\mathbf{x}^k - \mathbf{y}^k\| &\le \|\mathbf{x}^k - \bar{\mathbf{c}}\| + \|\mathbf{y}^k - (\mathbf{c}^k + \Delta\mathbf{c}^k)\| + \|\mathbf{c}^k + \Delta\mathbf{c}^k - \bar{\mathbf{c}}\| \\ &\le 2\|\mathbf{c}^k + \Delta\mathbf{c}^k - \bar{\mathbf{c}}\|. \end{aligned} \tag{25}$$

By (25) and Lemma 4.2, there exists a positive number ξ such that, for all k sufficiently large,

$$\|\mathbf{p}_i(\mathbf{x}^k) - \mathbf{p}_i(\mathbf{y}^k)\| \le \xi\|\mathbf{x}^k - \mathbf{y}^k\| \quad \text{and} \quad \|\mathbf{q}_i(\mathbf{x}^k) - \mathbf{q}_i(\mathbf{y}^k)\| \le \xi\|\mathbf{x}^k - \mathbf{y}^k\|, \quad 1 \le i \le n. \tag{26}$$

Since $A(\cdot)$ has distinct singular values $\{\sigma_l(A(\cdot))\}_{l=1}^n$ at $\mathbf{x}^k, \mathbf{y}^k$, we have [41]

$$\frac{\partial\sigma_l(\mathbf{x}^k)}{\partial c_p} = \mathbf{p}_l^T(\mathbf{x}^k)\frac{\partial A(\mathbf{x}^k)}{\partial c_p}\mathbf{q}_l(\mathbf{x}^k), \quad \frac{\partial\sigma_l(\mathbf{y}^k)}{\partial c_p} = \mathbf{p}_l^T(\mathbf{y}^k)\frac{\partial A(\mathbf{y}^k)}{\partial c_p}\mathbf{q}_l(\mathbf{y}^k), \quad 1 \le l, p \le n.$$

This, together with (24), (25), (26), and Lemma 4.1, implies that, for all k sufficiently large,

$$\begin{aligned} &\|\mathbf{g}'(\bar{\mathbf{c}}; \Delta\mathbf{c}^k) - \mathbf{g}'(\mathbf{c}^k + \Delta\mathbf{c}^k; \Delta\mathbf{c}^k)\| = \|V(\bar{\mathbf{c}})\Delta\mathbf{c}^k - V(\mathbf{c}^k + \Delta\mathbf{c}^k)\Delta\mathbf{c}^k\| \\ \le\ & \|V(\bar{\mathbf{c}}) - \mathbf{g}'(\mathbf{x}^k)\|\|\Delta\mathbf{c}^k\| + \|V(\mathbf{c}^k + \Delta\mathbf{c}^k) - \mathbf{g}'(\mathbf{y}^k)\|\|\Delta\mathbf{c}^k\| + \|\mathbf{g}'(\mathbf{x}^k)\Delta\mathbf{c}^k - \mathbf{g}'(\mathbf{y}^k)\Delta\mathbf{c}^k\| \\ \le\ & \sqrt{n}\max_{1\le j\le n}\Big|\sum_{l=1}^{j}\{\mathbf{p}_l^T(\mathbf{x}^k)(\sum_{p=1}^{n}(\Delta\mathbf{c}^k)_pA_p)\mathbf{q}_l(\mathbf{x}^k) - \mathbf{p}_l^T(\mathbf{y}^k)(\sum_{p=1}^{n}(\Delta\mathbf{c}^k)_pA_p)\mathbf{q}_l(\mathbf{y}^k)\}\Big| \\ &\quad +2\|\mathbf{c}^k + \Delta\mathbf{c}^k - \bar{\mathbf{c}}\|\|\Delta\mathbf{c}^k\| \\ \le\ & 2\|\mathbf{c}^k + \Delta\mathbf{c}^k - \bar{\mathbf{c}}\|\|\Delta\mathbf{c}^k\| \\ &+\sqrt{n}\max_{1\le j\le n}\sum_{l=1}^{j}|\mathbf{p}_l^T(\mathbf{x}^k)(\sum_{p=1}^{n}(\Delta\mathbf{c}^k)_pA_p)\mathbf{q}_l(\mathbf{x}^k) - \mathbf{p}_l^T(\mathbf{y}^k)(\sum_{p=1}^{n}(\Delta\mathbf{c}^k)_pA_p)\mathbf{q}_l(\mathbf{x}^k)| \\ &+\sqrt{n}\max_{1\le j\le n}\sum_{l=1}^{j}|\mathbf{p}_l^T(\mathbf{y}^k)(\sum_{p=1}^{n}(\Delta\mathbf{c}^k)_pA_p)\mathbf{q}_l(\mathbf{x}^k) - \mathbf{p}_l^T(\mathbf{y}^k)(\sum_{p=1}^{n}(\Delta\mathbf{c}^k)_pA_p)\mathbf{q}_l(\mathbf{y}^k)| \\ \le\ & 2\|\mathbf{c}^k + \Delta\mathbf{c}^k - \bar{\mathbf{c}}\|\|\Delta\mathbf{c}^k\| + \sqrt{n}\max_{1\le j\le n}\sum_{l=1}^{j}\|\mathbf{p}_l(\mathbf{x}^k) - \mathbf{p}_l(\mathbf{y}^k)\|\|\sum_{p=1}^{n}(\Delta\mathbf{c}^k)_pA_p\| \end{aligned}$$

$$
\begin{aligned}
& +\sqrt{n}\max_{1\le j\le n}\sum_{l=1}^{j}\|\sum_{p=1}^{n}(\Delta\mathbf{c}^k)_p A_p\|\|\mathbf{q}_l(\mathbf{x}^k)-\mathbf{q}_l(\mathbf{y}^k)\| \\
\le\ & 2\|\mathbf{c}^k+\Delta\mathbf{c}^k-\bar{\mathbf{c}}\|\|\Delta\mathbf{c}^k\| + 2\xi\zeta\sqrt{n}\max_{1\le j\le n}\sum_{l=1}^{j}\|\mathbf{c}^k+\Delta\mathbf{c}^k-\bar{\mathbf{c}}\|\|\Delta\mathbf{c}^k\| \\
=\ & 2(1+\xi\zeta n\sqrt{n})\|\mathbf{c}^k+\Delta\mathbf{c}^k-\bar{\mathbf{c}}\|\|\Delta\mathbf{c}^k\| \le 2(1+\xi\zeta n\sqrt{n})\|\mathbf{z}^k+\mathbf{d}^k-\bar{\mathbf{z}}\|\|\mathbf{d}^k\|. \qquad (27)
\end{aligned}
$$

We further derive from (11) and (27) that, for all k sufficiently large,

$$
\begin{aligned}
& \|\omega'(\bar{\mathbf{z}};\mathbf{d}^k)-\omega'(\mathbf{z}^k+\mathbf{d}^k;\mathbf{d}^k)\| \\
=\ & \|\mathbf{w}(\bar{\mathbf{z}})^T\mathbf{w}'(\bar{\mathbf{z}};\mathbf{d}^k)-\mathbf{w}(\mathbf{z}^k+\mathbf{d}^k)^T\mathbf{w}'(\mathbf{z}^k+\mathbf{d}^k;\mathbf{d}^k)\| \\
\le\ & \|\mathbf{w}(\bar{\mathbf{z}})-\mathbf{w}(\mathbf{z}^k+\mathbf{d}^k)\|\|\mathbf{w}'(\bar{\mathbf{z}};\mathbf{d}^k)\| + \|\mathbf{w}(\mathbf{z}^k+\mathbf{d}^k)\|\|\mathbf{w}'(\bar{\mathbf{z}};\mathbf{d}^k)-\mathbf{w}'(\mathbf{z}^k+\mathbf{d}^k;\mathbf{d}^k)\| \\
\le\ & L\|\mathbf{z}^k+\mathbf{d}^k-\bar{\mathbf{z}}\|L\|\mathbf{d}^k\| + \|\mathbf{w}(\mathbf{z}^k+\mathbf{d}^k)\|\|\mathbf{g}'(\bar{\mathbf{c}};\Delta\mathbf{c}^k)-\mathbf{g}'(\mathbf{c}^k+\Delta\mathbf{c}^k;\Delta\mathbf{c}^k)\| \\
& +\|\mathbf{w}(\mathbf{z}^k+\mathbf{d}^k)\|\|(\epsilon^k+\Delta\epsilon^k-0)\Delta\mathbf{c}^k+\Delta\epsilon^k(\mathbf{c}^k+\Delta\mathbf{c}^k-\bar{\mathbf{c}})\| \\
\le\ & L^2\|\mathbf{z}^k+\mathbf{d}^k-\bar{\mathbf{z}}\|\|\mathbf{d}^k\| + \|\mathbf{w}(\mathbf{z}^k+\mathbf{d}^k)\|\|\mathbf{g}'(\bar{\mathbf{c}};\Delta\mathbf{c}^k)-\mathbf{g}'(\mathbf{c}^k+\Delta\mathbf{c}^k;\Delta\mathbf{c}^k)\| \\
& \quad +2\|\mathbf{w}(\mathbf{z}^k+\mathbf{d}^k)\|\|\mathbf{z}^k+\mathbf{d}^k-\bar{\mathbf{z}}\|\|\mathbf{d}^k\| \\
=\ & L_1(\mathbf{z}^k,\mathbf{d}^k)\|\mathbf{z}^k+\mathbf{d}^k-\bar{\mathbf{z}}\|\|\mathbf{d}^k\|. \qquad (28)
\end{aligned}
$$

Therefore, the estimate (19) holds.

We now verify (20). By using the strong semismoothness of $\mathbf{w}$ and (22), it follows that for all k sufficiently large,

$$
\|\mathbf{w}'(\mathbf{z}^k;\mathbf{z}^k-\bar{\mathbf{z}})-\mathbf{w}'(\bar{\mathbf{z}};\mathbf{z}^k-\bar{\mathbf{z}})\| = \|V_k(\mathbf{z}^k-\bar{\mathbf{z}})-\mathbf{w}'(\bar{\mathbf{z}};\mathbf{z}^k-\bar{\mathbf{z}})\| = O(\|\mathbf{z}^k-\bar{\mathbf{z}}\|^2). \qquad (29)
$$

Also, by the hypothesis, one has

$$
\|\mathbf{w}(\mathbf{z}^k)+\mathbf{w}'(\mathbf{z}^k;\mathbf{d}^k)\| \le \eta_k\|\mathbf{w}(\mathbf{z}^k)\| \le \|\mathbf{w}(\mathbf{z}^k)\|^2 = \|\mathbf{w}(\mathbf{z}^k)-\mathbf{w}(\bar{\mathbf{z}})\|^2 = O(\|\mathbf{z}^k-\bar{\mathbf{z}}\|^2). \qquad (30)
$$

Using the strong semismoothness of $\mathbf{w}$, (22), (29), and (30), we obtain, for all k sufficiently large,

$$
\begin{aligned}
& \|\mathbf{z}^k+\mathbf{d}^k-\bar{\mathbf{z}}\| \\
\le\ & \|V_k^{-1}\|\|\mathbf{w}(\mathbf{z}^k)-\mathbf{w}(\bar{\mathbf{z}})-\mathbf{w}'(\bar{\mathbf{z}};\mathbf{z}^k-\bar{\mathbf{z}})\| + \|V_k^{-1}\|\|\mathbf{w}(\mathbf{z}^k)+V_k\mathbf{d}^k\| \\
& \quad +\|V_k^{-1}\|\|\mathbf{w}'(\mathbf{z}^k;\mathbf{z}^k-\bar{\mathbf{z}})-\mathbf{w}'(\bar{\mathbf{z}};\mathbf{z}^k-\bar{\mathbf{z}})\| \\
=\ & O(\|\mathbf{z}^k-\bar{\mathbf{z}}\|^2) + \|V_k^{-1}\|\|\mathbf{w}(\mathbf{z}^k)+\mathbf{w}'(\mathbf{z}^k;\mathbf{d}^k)\| + O(\|\mathbf{z}^k-\bar{\mathbf{z}}\|^2) \\
=\ & O(\|\mathbf{z}^k-\bar{\mathbf{z}}\|^2),
\end{aligned}
$$

which verifies (20).

Finally, we establish (21). Since $\mathbf{w}$ is strongly semismooth, by Lemma 2.2 and (20), one has, for any $\delta \in (0, 1/2)$ and for all k sufficiently large,

$$
\begin{aligned}
& \|\mathbf{w}(\mathbf{z}^k+\mathbf{d}^k)-\mathbf{w}'(\bar{\mathbf{z}};\mathbf{z}^k+\mathbf{d}^k-\bar{\mathbf{z}})\| \\
=\ & \|\mathbf{w}(\mathbf{z}^k+\mathbf{d}^k)-\mathbf{w}(\bar{\mathbf{z}})-\mathbf{w}'(\bar{\mathbf{z}};\mathbf{z}^k+\mathbf{d}^k-\bar{\mathbf{z}})\| \\
\le\ & \delta\|\mathbf{z}^k+\mathbf{d}^k-\bar{\mathbf{z}}\|. \qquad (31)
\end{aligned}
$$

By (20), for all k sufficiently large,

$$\|\mathbf{z}^k + \mathbf{d}^k - \bar{\mathbf{z}}\| \leq \frac{\delta}{1 + 2\kappa(1+\eta)(L+1)} \|\mathbf{z}^k - \bar{\mathbf{z}}\| \leq \delta \|\mathbf{z}^k - \bar{\mathbf{z}}\|. \tag{32}$$

It follows from (31) and (32) that, for all k sufficiently large,

$$\begin{aligned} \|\mathbf{w}(\mathbf{z}^k + \mathbf{d}^k)\| &\leq \|\mathbf{w}'(\bar{\mathbf{z}}; \mathbf{z}^k + \mathbf{d}^k - \bar{\mathbf{z}})\| + \|\mathbf{w}(\mathbf{z}^k + \mathbf{d}^k) - \mathbf{w}'(\bar{\mathbf{z}}; \mathbf{z}^k + \mathbf{d}^k - \bar{\mathbf{z}})\| \\ &\leq L\|\mathbf{z}^k + \mathbf{d}^k - \bar{\mathbf{z}}\| + \delta\|\mathbf{z}^k + \mathbf{d}^k - \bar{\mathbf{z}}\| \\ &\leq (L+\delta)\|\mathbf{z}^k + \mathbf{d}^k - \bar{\mathbf{z}}\| \\ &\leq \frac{(L+\delta)\delta}{1 + 2\kappa(1+\eta)(L+1)} \|\mathbf{z}^k - \bar{\mathbf{z}}\|. \end{aligned} \tag{33}$$

Also, by (18) and (32), we get, for all k sufficiently large,

$$\|\mathbf{z}^k - \bar{\mathbf{z}}\| \leq \|\mathbf{d}^k\| + \|\mathbf{z}^k + \mathbf{d}^k - \bar{\mathbf{z}}\| \leq \kappa(1+\eta)\|\mathbf{w}(\mathbf{z}^k)\| + \delta\|\mathbf{z}^k - \bar{\mathbf{z}}\|,$$

which implies that, for all k sufficiently large,

$$\|\mathbf{z}^k - \bar{\mathbf{z}}\| \leq \frac{\kappa(1+\eta)}{1-\delta} \|\mathbf{w}(\mathbf{z}^k)\|. \tag{34}$$

By (33) and (34), we obtain, for all k sufficiently large,

$$\begin{aligned} \|\mathbf{w}(\mathbf{z}^k + \mathbf{d}^k)\| &\leq \frac{(L+\delta)\delta}{1 + 2\kappa(1+\eta)(L+1)} \|\mathbf{z}^k - \bar{\mathbf{z}}\| \\ &\leq \frac{(L+\delta)\delta}{1 + 2\kappa(1+\eta)(L+1)} \frac{\kappa(1+\eta)}{1-\delta} \|\mathbf{w}(\mathbf{z}^k)\| \\ &\leq \frac{2\kappa(1+\eta)(L+1)\delta}{1 + 2\kappa(1+\eta)(L+1)} \|\mathbf{w}(\mathbf{z}^k)\| \\ &\leq \delta\|\mathbf{w}(\mathbf{z}^k)\|. \end{aligned}$$

This completes the proof of (21).

□

We now offer a lower and an upper bound for the sequence $\{\omega'(\mathbf{z}^k; \mathbf{d}^k)\}$ generated by Algorithm I.

Lemma 4.6 *Let $\{\mathbf{z}^k\}$ and $\{\mathbf{d}^k\}$ be generated by Algorithm I. Then one has*

$$-\omega'(\mathbf{z}^k; \mathbf{d}^k) \leq (1+\eta)\|\mathbf{w}(\mathbf{z}^k)\|^2 = 2(1+\eta)\omega(\mathbf{z}^k), \tag{35}$$

$$\omega'(\mathbf{z}^k; \mathbf{d}^k) \leq (\eta - 1)\|\mathbf{w}(\mathbf{z}^k)\|^2. \tag{36}$$

Proof: It follows from (11) and (13) that

$$\begin{aligned}
-\omega'(\mathbf{z}^k;\mathbf{d}^k) &= -\mathbf{w}(\mathbf{z}^k)^T\mathbf{w}'(\mathbf{z}^k;\mathbf{d}^k) = \langle -\mathbf{w}(\mathbf{z}^k), \mathbf{w}'(\mathbf{z}^k;\mathbf{d}^k)\rangle \\
&= \langle -\mathbf{w}(\mathbf{z}^k), \mathbf{w}'(\mathbf{z}^k;\mathbf{d}^k) + \mathbf{w}(\mathbf{z}^k) - \mathbf{w}(\mathbf{z}^k)\rangle \\
&\leq \|\mathbf{w}'(\mathbf{z}^k;\mathbf{d}^k) + \mathbf{w}(\mathbf{z}^k)\|\|\mathbf{w}(\mathbf{z}^k)\| + \|\mathbf{w}(\mathbf{z}^k)\|^2 \\
&\leq (1+\eta_k)\|\mathbf{w}(\mathbf{z}^k)\|^2 \leq (1+\eta)\|\mathbf{w}(\mathbf{z}^k)\|^2 \\
&= 2(1+\eta)\omega(\mathbf{z}^k);
\end{aligned}$$

and

$$\begin{aligned}
\omega'(\mathbf{z}^k;\mathbf{d}^k) &= \mathbf{w}(\mathbf{z}^k)^T\mathbf{w}'(\mathbf{z}^k;\mathbf{d}^k) = \langle \mathbf{w}(\mathbf{z}^k), \mathbf{w}'(\mathbf{z}^k;\mathbf{d}^k)\rangle \\
&= \langle \mathbf{w}(\mathbf{z}^k), \mathbf{w}'(\mathbf{z}^k;\mathbf{d}^k) + \mathbf{w}(\mathbf{z}^k) - \mathbf{w}(\mathbf{z}^k)\rangle \\
&\leq \|\mathbf{w}'(\mathbf{z}^k;\mathbf{d}^k) + \mathbf{w}(\mathbf{z}^k)\|\|\mathbf{w}(\mathbf{z}^k)\| - \|\mathbf{w}(\mathbf{z}^k)\|^2 \\
&\leq (\eta_k - 1)\|\mathbf{w}(\mathbf{z}^k)\|^2 \leq (\eta-1)\|\mathbf{w}(\mathbf{z}^k)\|^2.
\end{aligned}$$

□

The following theorem gives the global convergence of Algorithm I.

Theorem 4.7 *Let* $\{\mathbf{z}^k\}$ *be the sequence generated by Algorithm I. Suppose that the level set* $\{\mathbf{z} \in \mathbb{R}^{n+1} : \|\mathbf{w}(\mathbf{z})\| \leq \|\mathbf{w}(\mathbf{z}^0)\|\}$ *is bounded and each generalized Newton equation* (12) *can be solved inexactly such that the conditions* (13) *and* (14) *are satisfied. Assume that* $\mathbf{w}(\mathbf{z}^k) \neq \mathbf{0}$ *for all* k*. Then the following conclusions* (a)–(d) *hold:*

(a) $\|\mathbf{w}(\mathbf{z}^{k+1})\| < \|\mathbf{w}(\mathbf{z}^k)\|$,

(b) *the sequence* $\{\mathbf{z}^k\}$ *is bounded,*

(c) *if* $\bar{\mathbf{z}}$ *is any accumulation point of* $\{\mathbf{z}^k\}$*, Assumption 4.3 holds, and* $\mathcal{D}$ *is dense in* $\mathbb{R}^n$*, then* $\mathbf{w}(\bar{\mathbf{z}}) = \mathbf{0}$.

(d) *if* $\liminf \alpha_k > 0$*, then any accumulation point* $\bar{\mathbf{z}}$ *of* $\{\mathbf{z}^k\}$ *is such that* $\mathbf{w}(\bar{\mathbf{z}}) = \mathbf{0}$.

Proof: By the hypothesis, it is obvious that the conclusions (a) and (b) hold. In the following, we verify the conclusions (c) and (d). Since the sequence $\{\omega(\mathbf{z}^k) \geq 0\}$ is strictly monotone decreasing. Hence, $\{\omega(\mathbf{z}^k)\}$ is convergent and

$$\lim_{k\to\infty} \left(\omega(\mathbf{z}^k) - \omega(\mathbf{z}^{k+1})\right) = 0. \tag{37}$$

Using (15) and (36), we have

$$\omega(\mathbf{z}^k) - \omega(\mathbf{z}^{k+1}) \geq -\lambda_1\alpha_k\omega'(\mathbf{z}^k;\mathbf{d}^k) \geq \lambda_1(1-\eta)\alpha_k\|\mathbf{w}(\mathbf{z}^k)\|^2 \geq 0. \tag{38}$$

Relations (37) and (38) imply that

$$\lim_{k\to\infty}(\alpha_k\|\mathbf{w}(\mathbf{z}^k)\|^2) = 0.$$

If $\liminf \alpha_k > 0$, then, any accumulation point $\bar{\mathbf{z}}$ of $\{\mathbf{z}^k\}$ is a solution to $\mathbf{w}(\bar{\mathbf{z}}) = \mathbf{0}$. Now, we assume that $\liminf \alpha_k = 0$. Let $\bar{\mathbf{z}}$ be an accumulation point of $\{\mathbf{z}^k\}$. By taking a subsequence if necessary, we may assume that $\{\mathbf{z}^k\}$ converges to $\bar{\mathbf{z}}$ and $\lim \alpha_k = 0$. Thus $\{\mathbf{w}(\mathbf{z}^k)\}$ converges to $\mathbf{w}(\bar{\mathbf{z}})$. From Lemma 4.5, we know that the sequence $\{\mathbf{d}^k\}$ is bounded for all k sufficiently large. Hence, $\{\mathbf{d}^k\}$ has an accumulation point $\bar{\mathbf{d}}$. Without loss of generality, we may assume that $\mathbf{d}^k$ converges to $\bar{\mathbf{d}}$.

As the proofs of (27) and (28), we can derive that, for all k sufficiently large,

$$\begin{cases} \|\omega'(\mathbf{z}^k + \alpha^k \mathbf{d}^k; \bar{\mathbf{d}}) - \omega'(\bar{\mathbf{z}}; \bar{\mathbf{d}})\| \le L_2(\mathbf{z}^k, \alpha^k \mathbf{d}^k) \|\mathbf{z}^k + \alpha^k \mathbf{d}^k - \bar{\mathbf{z}}\| \|\bar{\mathbf{d}}\|, \\ \|\omega'(\mathbf{z}^k; \bar{\mathbf{d}}) - \omega'(\bar{\mathbf{z}}; \bar{\mathbf{d}})\| \le L_3(\mathbf{z}^k) \|\mathbf{z}^k - \bar{\mathbf{z}}\| \|\bar{\mathbf{d}}\|, \end{cases} \tag{39}$$

where

$$L_2(\mathbf{z}^k, \alpha^k \mathbf{d}^k) := L^2 + 2(2 + \xi\zeta n\sqrt{n}) \|\mathbf{w}(\mathbf{z}^k + \alpha^k \mathbf{d}^k)\|, \ L_3(\mathbf{z}^k) := L^2 + 2(2 + \xi\zeta n\sqrt{n}) \|\mathbf{w}(\mathbf{z}^k)\|,$$

By the strong semismoothness of ω and Lemma 2.2,

$$\begin{cases} \|\omega'(\mathbf{z}^k + \alpha^k \mathbf{d}^k; \mathbf{d}^k) - \omega'(\mathbf{z}^k + \alpha^k \mathbf{d}^k; \bar{\mathbf{d}})\| \le W_1 \|\mathbf{d}^k - \bar{\mathbf{d}}\|, \\ \|\omega'(\mathbf{z}^k; \mathbf{d}^k) - \omega'(\mathbf{z}^k; \bar{\mathbf{d}})\| \le W_2 \|\mathbf{d}^k - \bar{\mathbf{d}}\|, \end{cases} \tag{40}$$

where W_1 and W_2 are, respectively, the Lipschitz constants of $\omega'(\mathbf{z}^k + \alpha^k \mathbf{d}^k; \cdot)$ and $\omega'(\mathbf{z}^k; \cdot)$ around $\bar{\mathbf{d}}$. One has by (39) and (40), for all k sufficiently large,

$$\begin{cases} \|\omega'(\mathbf{z}^k + \alpha^k \mathbf{d}^k; \mathbf{d}^k) - \omega'(\bar{\mathbf{z}}; \bar{\mathbf{d}})\| \le W_1 \|\mathbf{d}^k - \bar{\mathbf{d}}\| + L_2(\mathbf{z}^k, \alpha^k \mathbf{d}^k) \|\mathbf{z}^k + \alpha^k \mathbf{d}^k - \bar{\mathbf{z}}\| \|\bar{\mathbf{d}}\|, \\ \|\omega'(\mathbf{z}^k; \mathbf{d}^k) - \omega'(\bar{\mathbf{z}}; \bar{\mathbf{d}})\| \le W_2 \|\mathbf{d}^k - \bar{\mathbf{d}}\| + L_3(\mathbf{z}^k) \|\mathbf{z}^k - \bar{\mathbf{z}}\| \|\bar{\mathbf{d}}\|, \end{cases} \tag{41}$$

Relations (16), (35) and (36) imply that

$$\begin{cases} \omega'(\mathbf{z}^k + \alpha^k \mathbf{d}^k; \mathbf{d}^k) > \lambda_2 \omega'(\mathbf{z}^k; \mathbf{d}^k) \ge -\lambda_2(1+\eta) \|\mathbf{w}(\mathbf{z}^k)\|^2, \\ \omega'(\mathbf{z}^k; \mathbf{d}^k) \le -(1-\eta) \|\mathbf{w}(\mathbf{z}^k)\|^2. \end{cases} \tag{42}$$

By passing to the limit as $k \to \infty$, we have by (41) and (42),

$$-\lambda_2(1+\eta) \|\mathbf{w}(\bar{\mathbf{z}})\|^2 \le \omega'(\bar{\mathbf{z}}; \bar{\mathbf{d}}) \le -(1-\eta) \|\mathbf{w}(\bar{\mathbf{z}})\|^2$$

and hence

$$\{(1-\eta) - \lambda_2(1+\eta)\} \|\mathbf{w}(\bar{\mathbf{z}})\|^2 \le 0.$$

Therefore, $\mathbf{w}(\bar{\mathbf{z}}) = \mathbf{0}$ since $\lambda_2(1+\eta) < (1-\eta)$. □

Finally, we can establish the quadratic convergence of Algorithm I as follows.

Theorem 4.8 *Let $\bar{\mathbf{z}} \in \mathbb{R}^{n+1}$ be an accumulation point of the sequence $\{\mathbf{z}^k\}$ generated by Algorithm I and $\mathbf{w}(\bar{\mathbf{z}}) = \mathbf{0}$. Assume that $\|\mathbf{w}(\mathbf{z}^k)\| \neq 0$ for all k and each generalized Newton equation* (12) *can be solved inexactly such that the conditions* (13) *and* (14) *are satisfied. Suppose that Assumption 4.3 is satisfied and $\mathcal{D}$ is dense in $\mathbb{R}^n$. Then the sequence $\{\mathbf{z}^k\}$ converges to $\bar{\mathbf{z}}$ quadratically and α_k eventually becomes* 1.

Proof: By Lemma 4.5, for all k sufficiently large,

$$\|\mathbf{z}^k + \mathbf{d}^k - \bar{\mathbf{z}}\| = O(\|\mathbf{z}^k - \bar{\mathbf{z}}\|^2). \tag{43}$$

Let $\gamma = \kappa(1+\eta) > 0$. By Lemma 4.5 again, for all k sufficiently large,

$$\begin{aligned} \|\mathbf{z}^k - \bar{\mathbf{z}}\| &\leq \|\mathbf{z}^k + \mathbf{d}^k - \bar{\mathbf{z}}\| + \|\mathbf{d}^k\| \\ &\leq \|\mathbf{z}^k + \mathbf{d}^k - \bar{\mathbf{z}}\| + \kappa(1+\eta)\|\mathbf{w}(\mathbf{z}^k)\| \\ &= \|\mathbf{z}^k + \mathbf{d}^k - \bar{\mathbf{z}}\| + \gamma\|\mathbf{w}(\mathbf{z}^k)\|. \end{aligned} \tag{44}$$

By (43), there exists a sufficiently small scalar $\theta > 0$ such that, for all k sufficiently large,

$$\|\mathbf{z}^k + \mathbf{d}^k - \bar{\mathbf{z}}\| \leq \theta\|\mathbf{z}^k - \bar{\mathbf{z}}\|.$$

This, together with (44), implies that, for all k sufficiently large,

$$\|\mathbf{z}^k + \mathbf{d}^k - \bar{\mathbf{z}}\| \leq \frac{\theta\gamma}{1-\theta}\|\mathbf{w}(\mathbf{z}^k)\|.$$

Thus one has, for all k sufficiently large,

$$\|\mathbf{w}(\mathbf{z}^k + \mathbf{d}^k)\| = \|\mathbf{w}(\mathbf{z}^k + \mathbf{d}^k) - \mathbf{w}(\bar{\mathbf{z}})\| \leq L\|\mathbf{z}^k + \mathbf{d}^k - \bar{\mathbf{z}}\| \leq \frac{L\theta\gamma}{1-\theta}\|\mathbf{w}(\mathbf{z}^k)\|$$

or

$$\omega(\mathbf{z}^k + \mathbf{d}^k) \leq \Big(\frac{L\theta\gamma}{1-\theta}\Big)^2 \omega(\mathbf{z}^k), \tag{45}$$

where L is the Lipschitz constant of $\mathbf{w}$ at $\bar{\mathbf{z}}$. Furthermore, we can choose a sufficiently small θ satisfying the condition

$$1 - \Big(\frac{L\theta\gamma}{1-\theta}\Big)^2 > 2(1+\eta)\lambda_1.$$

Then, we have by (35) and (45), for all k sufficiently large,

$$\omega(\mathbf{z}^k + \mathbf{d}^k) - \omega(\mathbf{z}^k) \leq \bigg(\Big(\frac{L\theta\gamma}{1-\theta}\Big)^2 - 1\bigg)\omega(\mathbf{z}^k) \leq -2(1+\eta)\lambda_1\omega(\mathbf{z}^k) \leq \lambda_1\omega'(\mathbf{z}^k; \mathbf{d}^k). \tag{46}$$

On the other hand, using (43), we get, for all k sufficiently large,

$$\|\mathbf{z}^k + \mathbf{d}^k - \bar{\mathbf{z}}\| \leq \frac{\lambda_1(1-\eta)}{\lambda_1(1-\eta) + L_0\gamma^2}\|\mathbf{z}^k - \bar{\mathbf{z}}\|,$$

where $L_0 := L^2 + 2(2 + \xi\zeta n\sqrt{n})\|\mathbf{w}(\mathbf{z}^0)\|$. This, together with (44), implies that

$$\|\mathbf{z}^k + \mathbf{d}^k - \bar{\mathbf{z}}\| \leq \frac{\lambda_1(1-\eta)}{L_0\gamma}\|\mathbf{w}(\mathbf{z}^k)\|. \tag{47}$$

By the strong semismoothness of ω, Lemma 4.5, Theorem 4.7(a), (36), and (47), for all k sufficiently large,

$$\begin{aligned}
-\omega'(\mathbf{z}^k+\mathbf{d}^k;\mathbf{d}^k) &= \omega'(\bar{\mathbf{z}};\mathbf{d}^k)-\omega'(\mathbf{z}^k+\mathbf{d}^k;\mathbf{d}^k) \\
&\leq \|\omega'(\bar{\mathbf{z}};\mathbf{d}^k)-\omega'(\mathbf{z}^k+\mathbf{d}^k;\mathbf{d}^k)\| \\
&\leq L_1(\mathbf{z}^k,\mathbf{d}^k)\|\mathbf{z}^k+\mathbf{d}^k-\bar{\mathbf{z}}\|\|\mathbf{d}^k\| \\
&= \big(L^2+2(2+\xi\zeta n\sqrt{n})\|\mathbf{w}(\mathbf{z}^k+\mathbf{d}^k)\|\big)\|\mathbf{z}^k+\mathbf{d}^k-\bar{\mathbf{z}}\|\|\mathbf{d}^k\| \\
&\leq \big(L^2+2(2+\xi\zeta n\sqrt{n})\|\mathbf{w}(\mathbf{z}^k)\|\big)\|\mathbf{z}^k+\mathbf{d}^k-\bar{\mathbf{z}}\|\|\mathbf{d}^k\| \\
&\leq L_0\frac{\lambda_1(1-\eta)}{L_0\gamma}\|\mathbf{w}(\mathbf{z}^k)\|\gamma\|\mathbf{w}(\mathbf{z}^k)\| \\
&= \lambda_1(1-\eta)\|\mathbf{w}(\mathbf{z}^k)\|^2 \leq -\lambda_1\omega'(\mathbf{z}^k;\mathbf{d}^k).
\end{aligned} \tag{48}$$

Notice that $\lambda_2>\lambda_1>0$. We have by (48), for all k sufficiently large,

$$\omega'(\mathbf{z}^k+\mathbf{d}^k;\mathbf{d}^k)\geq\lambda_1\omega'(\mathbf{z}^k;\mathbf{d}^k)>\lambda_2\omega'(\mathbf{z}^k;\mathbf{d}^k). \tag{49}$$

Combining (46) with (49) yields, for all k sufficiently large,

$$\alpha_k=1 \quad \text{and} \quad \mathbf{z}^{k+1}=\mathbf{z}^k+\mathbf{d}^k.$$

This, together with (43), completes the proof. □

Remark 4.9 *Theorem* 4.8 *shows that, under the conditions that Assumption* 4.3 *is satisfied and* $\mathcal{D}$ *is dense in* $\mathbb{R}^n$, $\bar{\mathbf{z}}$ *is a zero of* $\mathbf{w}$ *if and only if* $\{\mathbf{z}^k\}$ *converges to* $\bar{\mathbf{z}}$ *and* α_k *eventually becomes* 1. *In other words,* $\bar{\mathbf{z}}$ *is not a zero of* $\mathbf{w}$ *if and only if either* $\{\mathbf{z}^k\}$ *diverges or* $\lim_{k\to\infty}\alpha_k=0$. *This implies that the generalized Newton equation* (12) *may not give a globally descent direction for the merit function* ω *in* (10) *especially when* $\lim_{k\to\infty}\alpha_k=0$.

Remark 4.10 *In Theorem* 4.7(c) *and Theorem* 4.8, *we have assumed that* $\mathcal{D}$ *is dense in* $\mathbb{R}^n$, *i.e., all singular values of a real rectangular matrix are almost everywhere differentiable in* $\mathbb{R}^n$. *This is reasonable since the subset* $\mathbb{R}^n\backslash\mathcal{D}$ *is usually a null set. We can see the similar remark for eigenvalues of a real symmetric matrix as in* [15, pp. 646–648] *and* [38, p. 2365].

5 A hybrid method

We observe from Theorem 4.8 that, when $\mathbf{z}^k$ is sufficiently near a zero $\tilde{\mathbf{z}}$ of $\mathbf{w}$, the generalized Newton equation (12) exhibits a good local property: if Assumption 4.3 is satisfied and $\mathcal{D}$ is dense in $\mathbb{R}^n$, (12) is solvable and the convergence rate is quadratic. However, globally, the vector $\mathbf{d}^k$ generated by (12) is not necessarily a descent direction for the merit function ω in (10). Some globally convergent descent methods were provided in [17] for solving the nonlinear equations defined by locally Lipschitzian functions. To improve the practical effectiveness of Algorithm I, sparked by [17, 29], we give a hybrid method for solving $\mathbf{w}(\mathbf{z})=\mathbf{0}$, which combines the directional derivative-based generalized Newton equation (12) with an Armijo-like line search based on the merit function ω in (10). We will show that the hybrid method converges quadratically and globally (in the sense of computing a stationary point of the merit function ω).

Algorithm II: A hybrid Method

Step 0. *Given* $\eta \in (0, 1/2)$, $\rho \in (0, 1)$, $\lambda \in (0, 1-\eta)$, *and* $\bar{\epsilon} \in \mathbb{R}$. *Let* $\mathbf{z}^0 := (\epsilon^0, \mathbf{c}^0)$ *be the initial point, where* $\epsilon^0 := \bar{\epsilon}$ *and* $\mathbf{c}^0 \in \mathbb{R}^n$ *is arbitrary.* $k := 0$.

Step 1. *Solve* (12) *for* $\mathbf{d}^k := (\Delta\epsilon^k, \Delta\mathbf{c}^k) \in \mathbb{R} \times \mathbb{R}^n$ *such that the conditions* (13) *and* (14) *are satisfied.*

Step 2. *Compute the steplength* $\alpha_k = \rho^{l_k}$, *where* l_k *is the smallest nonnegative integer* l *satisfying the Armijo-like condition:*

$$\omega(\mathbf{z}^k + \rho^l \mathbf{d}^k) - \omega(\mathbf{z}^k) \leq -2\lambda\rho^l \omega(\mathbf{z}^k), \tag{50}$$

Set $\mathbf{z}^{k+1} := \mathbf{z}^k + \alpha_k \mathbf{d}^k$.

Step 3. *Replace* k *by* $k+1$ *and go to Step 1.*

We observe that, in Algorithm II, the Armijo-like rule (50) ensures that $\{\omega(\mathbf{z}^k)\}$ is strictly decreasing and is bounded below (by zero), i.e.,

$$0 < \omega(\mathbf{z}^{k+1}) \leq (1 - 2\lambda\alpha_k)\omega(\mathbf{z}^k) \leq \omega(\mathbf{z}^k).$$

In our numerical tests, we will report the numerical performance of Algorithm II instead of Algorithm I since Algorithm II is more effective and feasible in pratice.

As in [17], we can derive the follow lemma on the Armijo-like rule (50) in Algorithm II.

Lemma 5.1 *Let* $\eta \in (0, 1/2)$ *and* $\rho \in (0, 1)$. *Suppose that* $\omega(\mathbf{z}^k) \neq 0$ *for all* k. *If Equation* (12) *is solvable such that the conditions* (13) *and* (14) *are satisfied, then there exists a scalar* $\bar{\tau} > 0$ *such that for all* $\tau \in [0, \bar{\tau}]$,

$$\omega(\mathbf{z}^k + \tau\mathbf{d}^k) - \omega(\mathbf{z}^k) \leq -2\lambda\tau\omega(\mathbf{z}^k).$$

Proof: We show the conclusion by contradiction. Suppose there exists a sequence $\{\tau_i > 0\}$ such that $\{\tau_i\}$ converges to zero and

$$\omega(\mathbf{z}^k + \tau_i\mathbf{d}^k) - \omega(\mathbf{z}^k) > -2\lambda\tau_i\omega(\mathbf{z}^k) \quad \forall i.$$

Dividing both sides by τ_i and then passing to the limit $i \to \infty$, we get

$$\omega'(\mathbf{z}^k; \mathbf{d}^k) \geq -2\lambda\omega(\mathbf{z}^k). \tag{51}$$

On the other hand, by (11) and (13),

$$\begin{aligned}
\omega'(\mathbf{z}^k; \mathbf{d}^k) &= \mathbf{w}(\mathbf{z}^k)^T \mathbf{w}'(\mathbf{z}^k; \mathbf{d}^k) = \mathbf{w}(\mathbf{z}^k)^T \big(\mathbf{w}(\mathbf{z}^k) + \mathbf{w}'(\mathbf{z}^k; \mathbf{d}^k)\big) - \mathbf{w}(\mathbf{z}^k)^T \mathbf{w}(\mathbf{z}^k) \\
&\leq \|\mathbf{w}(\mathbf{z}^k)\| \|\mathbf{w}(\mathbf{z}^k) + \mathbf{w}'(\mathbf{z}^k; \mathbf{d}^k)\| - \|\mathbf{w}(\mathbf{z}^k)\|^2 \\
&\leq \eta_k \|\mathbf{w}(\mathbf{z}^k)\|^2 - \|\mathbf{w}(\mathbf{z}^k)\|^2 \\
&\leq \eta \|\mathbf{w}(\mathbf{z}^k)\|^2 - \|\mathbf{w}(\mathbf{z}^k)\|^2 \\
&\leq -2(1-\eta)\omega(\mathbf{z}^k).
\end{aligned} \tag{52}$$

By (51) and (52),

$$-2\lambda\omega(\mathbf{z}^k) \le -2(1-\eta)\omega(\mathbf{z}^k).$$

This is a contradiction by the choice of λ and the assumption that $\omega(\mathbf{z}^k) \neq 0$ for all k. □

We see from Lemma 5.1 that the Armijo-like rule (50) always holds for some nonnegative integer l_k under the assumption that each generalized Newton equation (12) is solvable such that the conditions (13) and (14) are satisfied.

On the global and quadratic convergence of Algorithm II, we have the following result.

Theorem 5.2 *Let $\bar{\mathbf{z}}$ be any accumulation point of the sequence $\{\mathbf{z}^k\}$ generated by Algorithm II. Suppose that the level set $\{\mathbf{z} \in \mathbb{R}^{n+1} : \|\mathbf{w}(\mathbf{z})\| \le \|\mathbf{w}(\mathbf{z}^0)\|\}$ is bounded and each generalized Newton equation* (12) *can be solved inexactly such that the conditions* (13) *and* (14) *are satisfied. Assume that $\|\mathbf{w}(\mathbf{z}^k)\| \neq 0$ for all k. Then the conclusions* (a)–(d) *hold:*

(a) $\|\mathbf{w}(\mathbf{z}^{k+1})\| < \|\mathbf{w}(\mathbf{z}^k)\|$,

(b) *the sequence $\{\mathbf{z}^k\}$ is bounded,*

(c) *if all elements in $\partial_B\mathbf{w}(\bar{\mathbf{z}})$ are nonsingular and $\mathcal{D}$ is dense in $\mathbb{R}^n$, then $\mathbf{w}(\bar{\mathbf{z}}) = \mathbf{0}$.*

(d) *if $\liminf \alpha_k > 0$, then $\mathbf{w}(\bar{\mathbf{z}}) = \mathbf{0}$.*

In addition, suppose that $\mathbf{w}(\bar{\mathbf{z}}) = \mathbf{0}$, all elements in $\partial_B\mathbf{w}(\bar{\mathbf{z}})$ are nonsingular and $\mathcal{D}$ is dense in $\mathbb{R}^n$. Then the sequence $\{\mathbf{z}^k\}$ converges to $\bar{\mathbf{z}}$ quadratically and α_k eventually becomes 1.

Proof: Obviously, the conclusions (a) and (b) hold.

If all elements in $\partial_B\mathbf{w}(\bar{\mathbf{z}})$ are nonsingular and $\mathcal{D}$ is dense in $\mathbb{R}^n$, we can show the conclusions (c) and (d) by following the similar proof of Theorem 4.7.

Suppose that $\mathbf{w}(\bar{\mathbf{z}}) = \mathbf{0}$, all elements in $\partial_B\mathbf{w}(\bar{\mathbf{z}})$ are nonsingular and $\mathcal{D}$ is dense in $\mathbb{R}^n$. Then, by Theorem 4.8, we can easily prove that $\{\mathbf{z}^k\}$ converges to $\bar{\mathbf{z}}$ quadratically and α_k eventually becomes 1. □

6 Numerical tests

In this section, we report some numerical tests to illustrate the effectiveness of Algorithm II for solving the ISVP. The tests were carried out in `MATLAB 7.10` running on a PC Intel Pentium IV of 3.0 GHz CPU.

In our numerical experiments, we set $\lambda = 10^{-4}$ and $\eta = 10^{-6}$. We implement the Armijo-like rule (50) by

$$\|\mathbf{w}(\mathbf{z}^{k+1})\| \le (1-2\lambda\alpha_k)^{\frac{1}{2}}\|\mathbf{w}(\mathbf{z}^k)\|.$$

For demonstration purpose, for Examples 6.1–6.5 below, we set the stopping criterion to be $\|\mathbf{w}(\mathbf{z}^k)\| - \|\mathbf{w}(\mathbf{z}^{k+1})\| \le 10^{-5}$ and the linear system (12) is solved by the TFQMR method [14] (we set the largest number of iterations in TFQMR to be 100). One may solve (12) by other iterative methods.

Example 6.1 *This is an inverse problem with distinct singular values* ($m = 7$ *and* $n = 4$)*. The test data is given as follows:*

$$A_0 = \begin{bmatrix} -0.1164 & 0.4062 & 1.7948 & 1.0345 \\ -0.9105 & 0.1345 & 0.6192 & -0.3270 \\ 0.6135 & -1.2327 & 0.8099 & -0.2056 \\ 1.8988 & -0.5718 & 2.2377 & -0.7730 \\ -0.6068 & -0.4616 & 1.9587 & -0.4333 \\ -0.3107 & -0.6635 & 0.1865 & -2.1145 \\ 0.7558 & 1.2255 & 0.7649 & -1.4273 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} -0.2576 & -0.2767 & 1.4645 & -0.9902 \\ -0.5142 & 0.4417 & 0.2714 & -1.3169 \\ -0.3264 & 0.6838 & 0.1268 & 1.2678 \\ 0.0355 & -1.0842 & -0.8186 & -1.5495 \\ 0.8300 & 1.1315 & -0.3695 & 1.6465 \\ 0.5172 & 0.5159 & 0.1272 & 1.0565 \\ -1.2333 & -1.2677 & 0.4796 & 1.4142 \end{bmatrix}, A_2 = \begin{bmatrix} 1.3894 & -0.3455 & -1.0254 & 0.9155 \\ 0.3137 & 1.2271 & 0.6192 & 0.4435 \\ -0.3831 & -0.2944 & -0.7685 & -0.0960 \\ 0.1327 & 1.8935 & 1.1052 & -1.9463 \\ 0.3461 & 1.7358 & -0.9458 & 0.2507 \\ 2.1518 & -0.9702 & -0.8940 & 0.2813 \\ 1.6980 & -0.1399 & 0.4265 & -0.3731 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1.1506 & 0.6360 & -0.0299 & -0.8582 \\ -0.2360 & 0.8084 & 0.9913 & -1.0706 \\ -0.3460 & 1.0914 & -1.0411 & 1.4576 \\ -0.6384 & 1.1379 & -0.5590 & -2.1814 \\ -0.4522 & 0.8223 & -1.1205 & -0.1578 \\ 0.6530 & 0.0416 & -1.4788 & -0.3093 \\ -0.2478 & 0.0248 & -1.3153 & 0.5597 \end{bmatrix}, A_4 = \begin{bmatrix} 1.2960 & 2.3942 & -1.0014 & -0.1239 \\ -0.1435 & 1.0879 & 1.6838 & 0.0109 \\ 1.4123 & -0.8940 & -0.2386 & -0.4911 \\ 0.3863 & -0.8844 & -2.1674 & -0.4211 \\ -0.5187 & -1.3002 & -0.7541 & 0.1669 \\ -0.0866 & 0.2277 & -1.2412 & 1.1240 \\ -1.3824 & 1.5823 & -1.4459 & -0.5267 \end{bmatrix},$$

$$\boldsymbol{\sigma}^* = (21.6179, 18.5490, 12.7146, 8.5357)^T, \quad \mathbf{c}^* = (1,2,3,4)^T.$$

We report our numerical results for different starting points: (a) $\mathbf{c}^0 = (-1,-1,-1,-1)^T$; (b) $\mathbf{c}^0 = (0,0,0,0)^T$; (c) $\mathbf{c}^0 = (10,10,10,10)^T$; (d) $\mathbf{c}^0 = (50,50,50,50)^T$; (e) $\mathbf{c}^0 = (100,100,100,100)^T$.

Example 6.2 *This is an inverse problem with distinct singular values* ($m = 5$ *and* $n = 4$)*. The test data is defined by:*

$$A_0 = \begin{bmatrix} -1.2099 & -1.2997 & 1.2996 & -0.2825 \\ -0.3411 & 1.4389 & -0.3705 & -1.3686 \\ -1.4465 & -0.2763 & -0.8841 & 0.0115 \\ -0.1558 & -0.3588 & -0.1188 & -0.4102 \\ -1.3954 & -0.0844 & -1.1884 & -0.6877 \end{bmatrix}, \quad A_i = I_m(:,i)I_n(:,i)^T, \quad i = 1,2,3,4,$$

$$\boldsymbol{\sigma}^* = (5.1602, 4.5999, 3.2861, 1.0943)^T, \quad \mathbf{c}^* = (1,2,-3,5)^T,$$

where $I_n(:,i)$ *denotes the* i*-th column of the identity matrix* I_n*. We report our numerical results for different starting points:* (a) $\mathbf{c}^0 = (0,0,0,0)^T$; (b) $\mathbf{c}^0 = (1,1,1,1)^T$; (c) $\mathbf{c}^0 = (10,10,10,10)^T$; (d) $\mathbf{c}^0 = (50,50,50,50)^T$; (e) $\mathbf{c}^0 = (10,-20,30,-50)^T$.

Example 6.3 *This is an inverse problem with multiple singular values ($m = 6$ and $n = 4$). The test data is defined by:*

$$A_0 = 0,$$

$$A_1 = \begin{bmatrix} 0.1347 & -1.0583 & -0.2869 & -0.1568 \\ -0.1734 & 0 & 0 & 0 \\ 0.3882 & 0 & 0 & 0 \\ -0.1521 & 0 & 0 & 0 \\ -0.5465 & 0 & 0 & 0 \\ 0.7477 & 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -0.1280 & -0.2836 & -0.1740 \\ 0 & 0.4002 & 0 & 0 \\ 0 & 0.2234 & 0 & 0 \\ 0 & 0.1311 & 0 & 0 \\ 0 & -0.1393 & 0 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0524 & -0.8364 \\ 0 & 0 & -0.0880 & 0 \\ 0 & 0 & 1.4272 & 0 \\ 0 & 0 & -0.1117 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.2905 \\ 0 & 0 & 0 & 0.0904 \\ 0 & 0 & 0 & 0.6021 \end{bmatrix},$$

$$\boldsymbol{\sigma}^* = (2, 1, 1, 1)^T, \quad \mathbf{c}^* = (1, 1, 1, 1)^T.$$

We report our numerical results for different starting points: (a) $\mathbf{c}^0 = (0, 0, 0, 0)^T$; (b) $\mathbf{c}^0 = (-3, 2, -4, 7)^T$; (c) $\mathbf{c}^0 = (5, 6, 7, 8)^T$; (d) $\mathbf{c}^0 = (11, 13, 15, 19)^T$; (e) $\mathbf{c}^0 = (100, 121, 155, 199)^T$.

Example 6.4 [2] *This is a Toeplitz-plus-Hankel inverse problem with distinct singular values ($m = n = 5$). Define*

$$A_0 = 0,$$

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix}, A_5 = \begin{bmatrix} 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{\sigma}^* = (4, 3, 2, 1, 0)^T, \quad \mathbf{c}^* = (0, 1.5388, 0.6882, 0.3633, 0.1625)^T.$$

We report our numerical results for different starting points: (a) $\mathbf{c}^0 = (0, 0, 0, 0, 0)^T$; (b) $\mathbf{c}^0 = (1, 1, 1, 1, 1)^T$; (c) $\mathbf{c}^0 = (8, 5, 8, 5, 8)^T$; (d) $\mathbf{c}^0 = (20, 17, 15, 12, 11)^T$; (e) $\mathbf{c}^0 = (37, 43, 48, 50, 60)^T$.

Example 6.5 [2] *This is a Toeplitz-plus-Hankel inverse problem with multiple singular values ($m = n = 5$). The basis matrices $\{A_i\}_{i=1}^5$ are defined as in Example 6.5 and*

$$\boldsymbol{\sigma}^* = (2, 2, 2, 1, 0)^T, \quad \mathbf{c}^* = (0, 1.1135, 0.1902, 0.1004, 0.1176)^T.$$

We report our numerical results for various starting points: (a) $\mathbf{c}^0 = (0,0,0,0,0)^T$*;* (b) $\mathbf{c}^0 = (1,1,1,1,1)^T$*;* (c) $\mathbf{c}^0 = (8,10,13,17,19)^T$*;* (d) $\mathbf{c}^0 = (30,27,25,22,21)^T$*;* (e) $\mathbf{c}^0 = (68,77,88,100,120)^T$*.*

Tables 1–10 display the numerical results for Examples 6.1–6.5, where `SP.`, `IT.`, `NF.`, $\|\mathbf{w}(\mathbf{z}^k)\|$, α_k, and $\kappa_2(\mathbf{g}'(\mathbf{c}^k)+\epsilon^k I_n) := \|\mathbf{g}'(\mathbf{c}^k)+\epsilon^k I_n\|\|(\mathbf{g}'(\mathbf{c}^k)+\epsilon^k I_n)^{-1}\|$ stand for the starting point, the number of Newton iterations and the number of function evaluations, the values of $\|\mathbf{w}(\mathbf{z}^k)\|$ at last 3 iterates of Algorithm II, the eventual value of the steplength α_k at the final iteration of Algorithm II, and the condition number of $\mathbf{g}'(\mathbf{c}^k)+\epsilon^k I_n$ at last 3 iterates of Algorithm II, where $A(\mathbf{c}^k)$ has distinct singular values, respectively. Here,"*" denotes that $\mathbf{g}'(\mathbf{c}^k)+\epsilon^k I_n$ could not be estimated at current iterate $\mathbf{c}^k$ since $A(\mathbf{c}^k)$ has multiple singular values.

We observe from Table Tables 1–10 that both Algorithm II with regularization (i.e., $\bar{\epsilon} \neq 0$) and Algorithm II without regularization (i.e., $\bar{\epsilon} = 0$) are convergent for different starting points. Furthermore, the values of $\kappa_2(\mathbf{g}'(\mathbf{c}^k)+\epsilon^k I_n)$ are not large and thus $\mathbf{g}'(\mathbf{c}^k)+\epsilon^k I_n$ are all nonsingular for all k sufficiently large where $A(\mathbf{c}^k)$ has distinct singular values and thus Assumption 4.3 is satisfied. The quadratic convergence occurs at the final stage of the iterations. This confirms our theoretical prediction. We also see that our algorithm converges to a solution $\bar{\mathbf{c}}$ of the ISVP, which is not necessarily equal to the original $\mathbf{c}^*$.

In our numerical tests, we observe that different values of $\bar{\epsilon}$ and ρ may yield different numbers of outer iterations. An interesting question is to find the optimal parameters $\bar{\epsilon}$ and ρ, which needs future study.

Table 1: Numerical results for Example 6.1

Alg. II with regularization							
`SP.`	$\bar{\epsilon}$	ρ	`IT.`	`NF.`	$\|\mathbf{w}(\mathbf{z}^k)\|$	α_k	$\kappa_2(\mathbf{g}'(\mathbf{c}^k)+\epsilon^k I_n)$
(a)	−0.9	0.99	7	43	$(4.6\times10^{-4}, 2.3\times10^{-8}, 1.6\times10^{-14})$	1.0	(37.8892, 37.8886, 37.8886)
(b)	−0.2	0.99	7	29	$(3.7\times10^{-3}, 8.6\times10^{-6}, 1.9\times10^{-11})$	1.0	(33.4332, 33.5040, 33.5041)
(c)	−0.1	0.98	10	53	$(6.4\times10^{-4}, 1.6\times10^{-7}, 1.1\times10^{-14})$	1.0	(33.9049, 33.9226, 33.9227)
(d)	−0.1	0.98	12	34	$(1.0\times10^{-3}, 2.4\times10^{-7}, 2.0\times10^{-14})$	1.0	(147.7322, 147.8332, 147.8332)
(e)	−0.1	0.98	6	7	$(1.0\times10^{-3}, 1.9\times10^{-7}, 2.4\times10^{-14})$	1.0	(23.4397, 15.23.4459, 23.4459)
Alg. II without regularization							
`SP.`	$\bar{\epsilon}$	ρ	`IT.`	`NF.`	$\|\mathbf{w}(\mathbf{z}^k)\|$	α_k	$\kappa_2(\mathbf{g}'(\mathbf{c}^k)+\epsilon^k I_n)$
(a)	0	0.99	25	224	$(1.9\times10^{-3}, 1.5\times10^{-6}, 6.8\times10^{-13})$	1.0	(33.4735, 33.5041, 33.5041)
(b)	0	0.99	8	51	$(8.4\times10^{-4}, 1.0\times10^{-6}, 3.1\times10^{-13})$	1.0	(33.4789, 33.5041, 33.5041)
(c)	0	0.98	11	54	$(1.3\times10^{-3}, 3.2\times10^{-6}, 2.1\times10^{-12})$	1.0	(387.1729, 386.7768, 386.7773)
(d)	0	0.98	8	9	$(5.9\times10^{-4}, 2.0\times10^{-9}, 4.2\times10^{-14})$	1.0	(51.5977, 51.5947, 51.5947)
(e)	0	0.98	8	9	$(1.4\times10^{-2}, 1.4\times10^{-6}, 1.4\times10^{-12})$	1.0	(51.6734, 51.5947, 51.5947)

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Table 2: Final values of $\mathbf{c}^k$ for Example 6.1

	Alg. II with regularization	Alg. II without regularization
SP.	$\bar{\mathbf{c}}$	$\bar{\mathbf{c}}$
(a)	$(-1.6138, 3.2168, -0.6228, 4.6591)$	$(-4.4120, -0.2317, -0.9087, -3.3053)$
(b)	$(-4.4120, -0.2317, -0.9087, -3.3053)$	$(-4.4120, -0.2317, -0.9087, -3.3053)$
(c)	$(4.7112, 2.1925, -5.3599, 4.8453)$	$(1.0946, 0.9030, 3.6816, 3.8788)$
(d)	$(0.8170, -0.7925, 4.2914, 3.7502)$	$(3.9373, -0.6586, 2.5320, 3.4000)$
(e)	$(1.2070, -0.9159, 1.3632, 5.1828)$	$(3.9373, -0.6586, 2.5320, 3.4000)$

Table 3: Numerical results for Example 6.2

Alg. II with regularization							
SP.	$\bar{\epsilon}$	ρ	IT.	NF.	$\|\mathbf{w}(\mathbf{z}^k)\|$	α_k	$\kappa_2(\mathbf{g}'(\mathbf{c}^k) + \epsilon^k I_n)$
(a)	-0.9	0.4	9	13	$(1.5 \times 10^{-5}, 3.9 \times 10^{-9}, 5.6 \times 10^{-15})$	1.0	$(15.7173, 15.7257, 15.7257)$
(b)	-0.9	0.4	8	11	$(1.9 \times 10^{-4}, 1.5 \times 10^{-8}, 3.2 \times 10^{-15})$	1.0	$(9.0796, 9.0793, 9.0793)$
(c)	0.1	0.96	8	21	$(1.1 \times 10^{-4}, 5.1 \times 10^{-9}, 4.1 \times 10^{-15})$	1.0	$(6.7536, 6.7538, 6.7538)$
(d)	-0.9	0.96	8	17	$(1.0 \times 10^{-4}, 1.9 \times 10^{-7}, 6.4 \times 10^{-13})$	1.0	$(16.5496, 16.6072, 16.6073)$
(e)	-0.1	0.99	5	6	$(2.3 \times 10^{-3}, 3.9 \times 10^{-6}, 1.5 \times 10^{-11})$	1.0	$(5.8986, 5.9103, 5.9103)$
Alg. II without regularization							
SP.	$\bar{\epsilon}$	ρ	IT.	NF.	$\|\mathbf{w}(\mathbf{z}^k)\|$	α_k	$\kappa_2(\mathbf{g}'(\mathbf{c}^k) + \epsilon^k I_n)$
(a)	0	0.4	8	12	$(3.8 \times 10^{-4}, 2.5 \times 10^{-6}, 1.2 \times 10^{-10})$	1.0	$(15.5148, 15.7243, 15.7257)$
(b)	0	0.4	6	12	$(2.6 \times 10^{-2}, 4.3 \times 10^{-6}, 3.0 \times 10^{-13})$	1.0	$(6.0137, 6.0154, 6.0154)$
(c)	0	0.96	19	119	$(1.6 \times 10^{-4}, 6.9 \times 10^{-8}, 2.5 \times 10^{-14})$	1.0	$(8.9459, 8.9483, 8.9483)$
(d)	0	0.96	21	201	$(2.8 \times 10^{-4}, 2.0 \times 10^{-7}, 1.1 \times 10^{-13})$	1.0	$(8.9440, 8.9483, 8.9483)$
(e)	0	0.99	9	52	$(7.0 \times 10^{-4}, 5.3 \times 10^{-8}, 2.5 \times 10^{-15})$	1.0	$(5.6568, 5.6571, 5.6571)$

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Table 4: Final values of $\mathbf{c}^k$ for Example 6.2

	Alg. II with regularization	Alg. II without regularization
SP.	$\bar{\mathbf{c}}$	$\bar{\mathbf{c}}$
(a)	$(-1.4792, -2.7890, -3.3801, -4.1488)$	$(-1.4792, -2.7890, -3.3801, -4.1488)$
(b)	$(0.9407, 1.6057, -3.1451, -4.3274)$	$(-2.7335, -0.2139, -1.8333, 5.2669)$
(c)	$(3.5784, -0.3461, 5.0644, 5.2830)$	$(-3.2677, 1.4269, -3.3568, 1.7424)$
(d)	$(-1.5039, -2.7937, -3.5671, -3.9496)$	$(-3.2677, 1.4269, -3.3568, 1.7424)$
(e)	$(5.6839, -0.4157, 5.1784, -2.5084)$	$(5.0938, -2.4771, 3.7809, -4.4343)$

Table 5: Numerical results for Example 6.3

Alg. II with regularization							
SP.	$\bar{\epsilon}$	ρ	IT.	NF.	$\|\mathbf{w}(\mathbf{z}^k)\|$	α_k	$\kappa_2(\mathbf{g}'(\mathbf{c}^k) + \epsilon^k I_n)$
(a)	-0.1	0.5	7	10	$(3.6 \times 10^{-4}, 1.6 \times 10^{-8}, 1.5 \times 10^{-15})$	1.0	$(65.1700, 96.9965, *)$
(b)	-0.1	0.5	7	8	$(5.3 \times 10^{-4}, 3.0 \times 10^{-7}, 1.3 \times 10^{-13})$	1.0	$(155.7666, 65.5588, *)$
(c)	-0.1	0.5	5	6	$(1.1 \times 10^{-3}, 5.3 \times 10^{-7}, 5.9 \times 10^{-14})$	1.0	$(70.2955, 92.9039, *)$
(d)	-0.1	0.5	8	11	$(9.1 \times 10^{-3}, 6.4 \times 10^{-6}, 8.7 \times 10^{-11})$	1.0	$(92.8700, 86.6766, *)$
(e)	-0.9	0.5	6	7	$(2.7 \times 10^{-5}, 1.2 \times 10^{-10}, 2.7 \times 10^{-15})$	1.0	$(95.5365, 140.8597, *)$
Alg. II without regularization							
SP.	$\bar{\epsilon}$	ρ	IT.	NF.	$\|\mathbf{w}(\mathbf{z}^k)\|$	α_k	$\kappa_2(\mathbf{g}'(\mathbf{c}^k) + \epsilon^k I_n)$
(a)	0	0.5	7	10	$(7.3 \times 10^{-4}, 5.9 \times 10^{-8}, 1.5 \times 10^{-15})$	1.0	$(65.1732, 80.7850, *)$
(b)	0	0.5	9	18	$(8.8 \times 10^{-4}, 3.4 \times 10^{-7}, 1.6 \times 10^{-13})$	1.0	$(71.3330, 725.9044, *)$
(c)	0	0.5	5	6	$(1.5 \times 10^{-4}, 6.4 \times 10^{-8}, 9.4 \times 10^{-15})$	1.0	$(277.3089, 65.2269, *)$
(d)	0	0.5	6	8	$(4.8 \times 10^{-4}, 5.5 \times 10^{-7}, 3.4 \times 10^{-14})$	1.0	$(87.4793, 67.0806, *)$
(e)	0	0.5	7	9	$(1.8 \times 10^{-4}, 4.6 \times 10^{-9}, 3.0 \times 10^{-15})$	1.0	$(141.7496, 68.2552, *)$

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Table 6: Final values of $\mathbf{c}^k$ for Example 6.3

	Alg. II with regularization	Alg. II without regularization
SP.	$\bar{\mathbf{c}}$	$\bar{\mathbf{c}}$
(a)	$(1.0000, 1.0000, 1.0000, 1.0000)$	$(1.0000, 1.0000, 1.0000, 1.0000)$
(b)	$(-1.0000, -1.0000, -1.0000, -1.0000)$	$(-1.0000, -1.0000, -1.0000, -1.0000)$
(c)	$(1.0000, 1.0000, 1.0000, 1.0000)$	$(1.0000, 1.0000, 1.0000, 1.0000)$
(d)	$(1.0000, 1.0000, 1.0000, 1.0000)$	$(1.0000, 1.0000, 1.0000, 1.0000)$
(e)	$(-1.0000, -1.0000, -1.0000, -1.0000)$	$(1.0000, 1.0000, 1.0000, 1.0000)$

Table 7: Numerical results for Example 6.4

Alg. II with regularization							
SP.	$\bar{\epsilon}$	ρ	IT.	NF.	$\Vert\mathbf{w}(\mathbf{z}^k)\Vert$	α_k	$\kappa_2(\mathbf{g}'(\mathbf{c}^k) + \epsilon^k I_n)$
(a)	-0.9	0.98	9	43	$(6.5 \times 10^{-3}, 4.7 \times 10^{-8}, 3.7 \times 10^{-15})$	1.0	$(38.8276, 38.8275, 38.8275)$
(b)	-0.9	0.5	8	15	$(2.8 \times 10^{-4}, 9.3 \times 10^{-8}, 1.6 \times 10^{-14})$	1.0	$(31.0204, 31.0194, 31.0194)$
(c)	-0.9	0.5	7	8	$(3.5 \times 10^{-3}, 2.9 \times 10^{-8}, 2.7 \times 10^{-15})$	1.0	$(38.8278, 38.8275, 38.8275)$
(d)	-0.9	0.5	5	6	$(4.2 \times 10^{-3}, 3.3 \times 10^{-6}, 3.1 \times 10^{-12})$	1.0	$(25.2340, 25.2419, 25.2419)$
(e)	-0.9	0.5	4	5	$(3.3 \times 10^{-3}, 1.5 \times 10^{-8}, 5.7 \times 10^{-15})$	1.0	$(38.8275, 38.8275, 38.8275)$
Alg. II without regularization							
SP.	$\bar{\epsilon}$	ρ	IT.	NF.	$\Vert\mathbf{w}(\mathbf{z}^k)\Vert$	α_k	$\kappa_2(\mathbf{g}'(\mathbf{c}^k) + \epsilon^k I_n)$
(a)	0	0.98	10	78	$(1.4 \times 10^{-2}, 9.7 \times 10^{-7}, 3.6 \times 10^{-15})$	1.0	$(38.8278, 38.8275, 38.8275)$
(b)	0	0.5	7	13	$(8.8 \times 10^{-4}, 3.4 \times 10^{-7}, 1.6 \times 10^{-13})$	1.0	$(28.5554, 28.5637, 28.5637)$
(c)	0	0.5	5	6	$(3.1 \times 10^{-3}, 2.7 \times 10^{-9}, 2.0 \times 10^{-15})$	1.0	$(38.8275, 38.8275, 38.8275)$
(d)	0	0.5	5	6	$(1.1 \times 10^{-5}, 2.5 \times 10^{-10}, 4.4 \times 10^{-15})$	1.0	$(28.5636, 28.5637, 28.5637)$
(e)	0	0.5	4	5	$(2.8 \times 10^{-3}, 1.9 \times 10^{-6}, 7.4 \times 10^{-13})$	1.0	$(28.5684, 28.5637, 28.5637)$

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Table 8: Final values of $\mathbf{c}^k$ for Example 6.4

	Alg. II with regularization	Alg. II without regularization
SP.	$\bar{\mathbf{c}}$	$\bar{\mathbf{c}}$
(a)	$(0.0000, 0.9960, 1.3764, -0.0898, 0.3249)$	$(-0.0000, 1.0858, -0.0898, 0.9062, 0.9960)$
(b)	$(0.9648, 0.2273, 0.2358, 0.7682, 1.0888)$	$(0.9583, 0.5146, 0.2372, 0.6007, 1.1027)$
(c)	$(-0.0000, 0.6710, 0.6155, -1.4662, -0.1453)$	$(0.0000, 0.4531, 1.2139, -0.5429, 1.0131)$
(d)	$(0.5535, -0.3526, -0.1610, -0.7867, -1.3650)$	$(0.9583, 1.1027, 0.6007, 0.2372, 0.5146)$
(e)	$(0.0000, 0.0555, 0.6155, -1.6115, -0.1453)$	$(0.9583, 0.5146, 0.2372, 0.6007, 1.1027)$

Table 9: Numerical results for Example 6.5

Alg. II with regularization							
SP.	$\bar{\epsilon}$	ρ	IT.	NF.	$\|\mathbf{w}(\mathbf{z}^k)\|$	α_k	$\kappa_2(\mathbf{g}'(\mathbf{c}^k) + \epsilon^k I_n)$
(a)	-0.8	0.8	5	7	$(3.2 \times 10^{-3}, 2.3 \times 10^{-8}, 1.5 \times 10^{-15})$	1.0	$(114.6501, 38.8649, *)$
(b)	-0.9	0.9	22	178	$(6.7 \times 10^{-2}, 1.1 \times 10^{-7}, 1.4 \times 10^{-14})$	1.0	$(38.8276, 38.9429, *)$
(c)	-0.9	0.98	26	257	$(2.1 \times 10^{-3}, 1.8 \times 10^{-9}, 2.9 \times 10^{-15})$	1.0	$(38.8547, 38.8378, *)$
(d)	0.1	0.96	6	28	$(3.1 \times 10^{-2}, 5.9 \times 10^{-6}, 7.6 \times 10^{-12})$	1.0	$(40.5730, 38.8277, *)$
(e)	-0.1	0.96	6	7	$(3.2 \times 10^{-2}, 4.6 \times 10^{-6}, 3.4 \times 10^{-12})$	1.0	$(38.9229, 38.8276, *)$
Alg. II without regularization							
SP.	$\bar{\epsilon}$	ρ	IT.	NF.	$\|\mathbf{w}(\mathbf{z}^k)\|$	α_k	$\kappa_2(\mathbf{g}'(\mathbf{c}^k) + \epsilon^k I_n)$
(a)	0	0.8	11	60	$(3.9 \times 10^{-3}, 1.1 \times 10^{-8}, 4.2 \times 10^{-15})$	1.0	$(38.8505, 38.8275, *)$
(b)	0	0.9	26	298	$(1.9 \times 10^{-3}, 9.6 \times 10^{-10}, 4.3 \times 10^{-15})$	1.0	$(38.8281, 38.8275, *)$
(c)	0	0.98	48	495	$(9.7 \times 10^{-3}, 1.4 \times 10^{-7}, 2.8 \times 10^{-15})$	1.0	$(39.0604, 38.8275, *)$
(d)	0	0.96	9	84	$(4.1 \times 10^{-2}, 8.3 \times 10^{-7}, 1.8 \times 10^{-15})$	1.0	$(38.8338, 38.8290, *)$
(e)	0	0.96	38	401	$(9.8 \times 10^{-3}, 1.9 \times 10^{-7}, 8.8 \times 10^{-15})$	1.0	$(52.0276, 38.8275, *)$

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Table 10: Final values of $\mathbf{c}^k$ for Example 6.5

	Alg. II with regularization	Alg. II without regularization
SP.	$\bar{\mathbf{c}}$	$\bar{\mathbf{c}}$
(a)	$(-0.0000, 1.1135, -0.1902, 0.1004, -0.1176)$	$(0.0000, 0.1902, 1.1135, -0.1176, -0.1004)$
(b)	$(0.0000, -0.6604, -0.3527, -0.6433, 0.5706)$	$(0.0000, -0.1902, 1.1135, 0.1176, -0.1004)$
(c)	$(-0.0000, -0.5706, 0.6433, 0.3527, 0.6604)$	$(-0.0000, -0.5706, 0.6433, 0.3527, 0.6604)$
(d)	$(0.0000, 0.1902, 1.1135, -0.1176, -0.1004)$	$(0.0000, -0.1176, 1.0409, -0.1902, -0.4082)$
(e)	$(-0.0000, -0.1902, -1.1135, 0.1176, 0.1004)$	$(0.0000, -0.6604, -0.3527, -0.6433, 0.5706)$

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