Robust and Minimum Norm Partial Quadratic Eigenvalue Assignment in Vibrating Systems: A New Optimization Approach

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Abstract

The partial quadratic eigenvalue assignment problem (PQEVAP) concerns reassigning a few undesired eigenvalues of a quadratic matrix pencil to suitably chosen locations and keeping the other large number of eigenvalues and eigenvectors unchanged (no spill-over). The problem naturally arises in controlling dangerous vibrations in structures by means of active feedback control design. For practical viability, the design must be robust, which requires that the norms of the feedback matrices and the condition number of the closed-loop eigenvectors are as small as possible. The problem of computing feedback matrices that satisfy the above two practical requirements is known as the Robust Partial Quadratic Eigenvalue Assignment Problem (RPQEVAP). In this paper, we formulate the RPQEVAP as an unconstrained minimization problem with the cost function involving the condition number of the closed-loop eigenvector matrix and two feedback norms. Since only a small number of eigenvalues of the open-loop quadratic pencil are computable using the state-of-the-art matrix computational techniques and/or measurable in a vibration laboratory, it is imperative that the problem is solved using these small number of eigenvalues and the corresponding eigenvectors. To this end, a class of the feedback matrices are obtained in parametric form, parameterized by a single parametric matrix, and the cost function and the required gradient formulas for the optimization problem are developed in terms of the small number of eigenvalues that are reassigned and their corresponding eigenvectors. The problem is solved directly in quadratic setting without transforming it to a standard first-order control problem and most importantly, the significant "no spill-over property" of the closed-loop eigenvalues and eigenvectors is established by means of a mathematical result. These features make the proposed method practically applicable even for very large structures. Results on numerical experiments show that the proposed method considerably reduces both feedback norms and the sensitivity of the closed-loop eigenvalues. A study on robustness of the system responses of the method under small perturbations show that the responses of the perturbed closed-loop system are compatible with perturbations.

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1 Introduction

Vibrating structures are distributed parameter systems [1, 2, 24]. However, for the sake of computational convenience, these are very often discretized into a system of second-order differential equations of the form [2, 24]:

$$M\ddot{\mathbf{x}}(t) + D\dot{\mathbf{x}}(t) + K\mathbf{x}(t) = 0, \tag{1}$$

where the three n-by-n matrices M, D, and K are, respectively, mass, damping and stiffness matrices. Very often they have special structures, such as, M, D, and K are symmetric, M is positive definite and diagonal or tridiagonal, and K is positive semidefinite and tridiagonal, etc.

By the separation of variables, $\mathbf{x}(t) = \mathbf{x}\, e^{\lambda t}$, where $\mathbf{x}$ is a constant vector, one can obtain the general solution of (1) in terms of the eigenvalues and eigenvectors $\{(\lambda_k, x_k)\}_{k=1}^{2n}$ of the quadratic matrix pencil

$$P(\lambda) = \lambda^2 M + \lambda D + K, \tag{2}$$

where $P(\lambda_k)x_k = 0$ for $k = 1, 2 \ldots, 2n$. The eigenvalues are related to natural frequencies and eigenvectors are mode shapes. Thus, the dynamics of (1) are governed by the eigenvalues and eigenvectors of $P(\lambda)$.

It is well-known that vibrating structures sometimes experience dangerous oscillations, such as resonance, causing partial or complete destruction of the structures. A widely used practice to control undesirable vibrations is to apply a passive control device, such as a shock absorber in a car. Though quite economic and easy to implement, such a passive device has severe practical limitations. These include its limited performance, changing the global dynamics, and others. On the other hand, because of the recent remarkable advances in sensors and actuators, the use of active vibration control force is becoming more popular. The active control devices are capable of overcoming the shortcomings of the passive devices. In active control strategy, vibrations are measured by sensors and then transmitted to a computer system where the required control force is computed in real-time and then applied to the structure by means of actuation.

Thus, the most important and challenging aspect of implementing an active control strategy is to efficiently and effectively compute the required feedback control force. Mathematically, the problem of computing the feedback control force can be described as follows:

Let a control force in the form $B\mathbf{u}(t)$, where B is an $n \times m$ control matrix and $\mathbf{u}(t)$ is the associated control vector, being applied to the structure, and suppose

$$\mathbf{u}(t) = F_1^T \dot{\mathbf{x}} + F_2^T \mathbf{x}(t), \tag{3}$$

where the n-by-m matrices F_1 and F_2 are called feedback matrices. Then the dynamics of the closed-loop system is governed by the eigenvalues and eigenvectors of the closed-loop pencil

$$P_c(\lambda) = \lambda^2 M + \lambda(D - BF_1^T) + (K - BF_2^T). \tag{4}$$

Since the resonance is caused when only a few natural frequencies become close to external frequencies, the problem of computing active feedback is to find two feedback matrices F_1 and

F_2 such that a few resonant eigenvalues, say $\lambda_1, \ldots, \lambda_p$ ($p \ll 2n$), are replaced by suitably chosen ones, $\mu_1, \ldots, \mu_p$, while the remaining $(2n - p)$ eigenvalues $\lambda_{p+1}, \ldots, \lambda_{2n}$ of $P(\lambda)$ and the associated eigenvectors, remain unchanged; that is, there will be **no spill-over** of the unassigned closed-loop eigenvalues and eigenvectors. *This property ensures that those eigenvalues that are not be specifically reassigned will not themselves become resonant or unstable.* The above problem is known as the **Partial Quadratic Eigenvalue Assignment Problem (PQEVAP)**.

Since there exists excellent numerical methods for solving eigenvalue assignment problems in standard first-order control system (see Datta [11]), including some for partial eigenvalue assignment problem, such as by Porter and Crossley [32], Datta and Sarkissian [19], and Datta and Saad [20], it is only natural to think of solving the PQEVAP by transforming the problem to a first-order control problem and then using a suitable first-order technique.

Unfortunately, there are severe computational drawbacks, associated with that approach. These include (i) requirement of inversion of a possibly ill-conditioned mass matrix, and (ii) loss of exploitable properties, such as the symmetry, positive definiteness, sparsity, etc. Furthermore, the existing eigenvalue methods are designed for small-order control problems and in order to use an existing method for a second order model, which are usually very large, the model order must be reduced considerably by using techniques of model reduction, such as the Guyan reduction technique (see Inman [23]). However, the existing model reduction techniques, including the popular Guyan reduction technique, may fail to produce even a few natural frequencies and mode shapes accurately (see Friswell and Mottershead [21]). Inspite of several attempts by numerical linear algebra researchers, effective numerical model reduction techniques that work exclusively in second-order model are rare. Similarly, the state-of-the-art **Independent Modal Space Control** (IMSC) approach (see Inman [23]), Meirovitch et al. [27, 28], also is not practical for the large and sparse PQEVAP. For open-loop decoupling, the IMSC approach needs the knowledge of the complete spectrum and eigenvectors of $P(\lambda)$, and for closed-loop decoupling, a stringent requirement of the number of sensors and actuators must be met (Inman [23], Meirovitch et al. [28]). On the other hand, the state-of-the-art computational technique, the Jacobi-Davidson method (see Datta [10], Tissuer and Meerbergen [38]) for the quadratic eigenproblem, is capable of computing only a few extremal eigenvalues and eigenvectors of a large quadratic pencil.

To meet these practical engineering and computational difficulties, several *"direct, partial-modal and no spill-over"* methods for the PQEVAP have been developed in recent years (Datta et al. [12, 13, 14, 15, 20], Ram and Elhay [34]), Chu [8]).

These methods (i) work directly in the second-order model, (ii) do not require any model reduction, and (iii) can be implemented with the knowledge of only a small number of eigenvalues which are to be reassigned and the corresponding eigenvectors and (iv) above all, *the no spill-over property is established by means of a mathematical theory based on the new orthogonality relations of the eigenvectors of the quadratic matrix pencil $P(\lambda)$, developed in these papers.* Datta and Sarkissian [18] and Datta, Ram and Sarkissian [16] have also developed a partial eigenvalue assignment method for gyroscopic distributed parameters systems that does not require discretizations of the original model.

Though these methods are suitable to compute a set of feedback matrices in a numerically desirable way, they do not address the problem of *robust control design.*

The robustness of the closed-loop eigenvalues and eigenvectors depends upon (i) norms of the feedback matrices, and (ii) the conditioning of the closed-loop eigenvectors (see Datta [11]). The small feedback norms guarantee small signals and therefore low cost, and low conditioning of the closed-loop eigenvalues assures that the closed-loop eigenvalues remain insensitive to small perturbations of the data. The latter is known as *Robust Partial Quadratic Eigenvalue Assignment Problem* (RPQEVAP).

Both the RPQEVAP and the problem of norm minimization are clearly optimization problems. To solve these problems in an optimization setting, some additional challenges must be met. These are:

- The feedback matrices must be expressed in some parametric forms so that the optimization problems can be solved by exploiting this parametric matrix.
- Gradient formulas must be developed using the knowledge of only these few eigenvalues and eigenvectors of $P(\lambda)$ that are computationally available or measurable.

There exist a few papers on robustness for the *complete eigenvalue assignment* in the first order control systems, such as those, by Keel, Fleming, and Bhattacharyya [26], Keel and Bhattacharyya [25], Cavin III and Bhattacharyya [7], Varga [39, 40], etc. However, work on robustness for quadratic partial eigenvalue assignment is rare. Only two papers that deals with the robustness of the PQEVAP have been published so far. They are: the paper by Qian and Xu [33] and the recent papers by Brahama and Datta [3, 4]. The method proposed by Qian and Xu is not an optimization-based and has limitations. Brahama and Datta, instead of minimizing the condition number of the closed-loop eigenvector matrix Y, $Cond_F(Y) = \|Y\|_F \|Y^{-1}\|_F$, minimized the expression $\|Y^T Y - I\|_F$. The rationale is that if the matrix Y is made close to an orthogonal matrix, then Y will be well-conditioned. *But it is desirable that* $Cond_F(Y)$ *itself is minimized* [5]. There also exists an earlier paper by Chu and Datta [9] that deals with robustness of the quadratic eigenvalue assignment but complete spectrum assignment is considered there. Another recent paper of Datta, Lin and Wang [15] considers robust partial assignment in a cubic pencil in an optimization setting that deals with robustness of vibrating structures with aerodynamics effects. The technique proposed there, however, requires knowledge of the complete spectrum and the associated eigenvectors for its implementation. Mottershead, Tehrani, and Ram [29] have also developed receptance-based method for desensitizing the closed-loop eigenvalues.

In this paper, a mathematically equivalent expression of $Cond_F(Y)$, namely, $\|Y\|_F^2 + \|Y^{-1}\|_F^2$ is minimized. This minimization problem then is combined with the problem of minimization of two feedback norms to simultaneously minimize the closed-loop condition number and feedback norms.

Numerical experiments are performed to:

- Study the performance of the proposed algorithm
- Compare the algorithm with the similar algorithm of Brahma and Datta
- Study the robustness of the system

The results of comparison of the proposed algorithm with the method by Brahma and Datta [4] show that the proposed algorithm is superior in its performances of both aspects of norms and the closed-loop condition number minimization.

Also, a comparison of the system responses of the original system with perturbed system, in which the matrix K goes under several small perturbations while the matrices $M,\ D,\ B$ are kept fixed, shows the responses remain insensitive to these perturbations and successfully approach to the steady state.

2 Notations and Assumptions:

$\{\lambda_1, \lambda_2, \ldots, \lambda_p;\ \lambda_{p+1}, \ldots, \lambda_{2n}\}$ − Eigenvalues of $P(\lambda)$.
$\Lambda_1 = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$ − Eigenvalue matrix of the eigenvalues that need to be reassigned.
X_1 = Matrix of eigenvectors corresponding to the eigenvalues in Λ_1.
$\Lambda_1^{'} = \text{diag}(\mu_1, \mu_2, \ldots, \mu_p)$ − Eigenvalue matrix of the eigenvalues that are to be assigned.
Y = The closed-loop eigenvector matrix.
Y_1 = Matrix of eigenvectors corresponding to the eigenvalues in $\Lambda_1^{'}$.
$\Lambda_2 = \text{diag}(\lambda_{p+1}, \lambda_{p+2}, \ldots, \lambda_{2n})$ − Eigenvalue matrix of the eigenvalues that are left invariant.
X_2 = Matrix of eigenvectors corresponding to the eigenvalues in Λ_2, which are also left invariant.
$\text{cond}_F(A) = \|A\|_F \|A^{-1}\|_F$ (Condition number of A in Frobenius norm).
A^H − Conjugate transpose of $A(A^H = (\bar{A})^T)$.
A^+ − The Moore-Penrose inverse of A.
$tr(A)$ − Trace of A.
$vec(\cdot)$ − The vec operator which vectorizes a matrix by stacking its columns.
$E \otimes F = \begin{bmatrix} e_{11}F & \ldots & e_{1n}F \\ \vdots & & \vdots \\ e_{m1}F & \ldots & e_{nn}F \end{bmatrix}$ − The Kronecker product of the two matrices $E = (e_{ij})_{m\times n}$ and F of order $s \times t$.

Throughout the paper, the following assumptions will be made:

(a) M, D, and K are symmetric and M is positive definite ($M > 0$),

(b) The control matrix B has full column rank,

(c) $\{\lambda_1, \lambda_2, \ldots, \lambda_p\} \cap \{\mu_1, \mu_2, \ldots, \mu_p\} = \emptyset$,

(d) $\{\lambda_1, \lambda_2, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_{2n}\} = \emptyset$,

(e) $(P(\lambda), B)$ is partially controllable with respect to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_p$; that is rank $(P(\lambda_i), B) = n,\ i = 1, 2, \ldots, p$.

3 Problem Statement

The problem of simultaneously minimizing the condition number of the closed-loop eigenvector matrix, Y and two feedback norms, can be stated in an optimization setting as follows:

$$\min J = \frac{1}{2}\alpha(\|Y\|_F^2 + \|Y^{-1}\|_F^2) + \frac{1}{2}(1-\alpha)(\|F_1\|_F^2 + \|F_2\|_F^2), \tag{5}$$

where $0 \leq \alpha \leq 1$.

Note that if $\alpha = 0$, then we have pure norm minimization problem and if $\alpha = 1$, then we have RPQEVAP.

4 Problem Solution

The solution to the problem comes in three sequential steps:

- A family of parametric feedback matrices, parameterized by a single parametric matrix Γ, is obtained.
- The closed-loop eigenvector matrix Y and its inverse are expressed in terms of the matrix Γ.
- Finally, the gradient expressions are derived with knowledge of only a small number of eigenvalues that need to be reassigned and the associated eigenvectors.

4.1 Parametric Expressions for Feedback Matrices F_1 and F_2

The derivation of the parametric expressions of F_1 and F_2 (given below) depends upon the following orthogonality relation of the quadratic pencil $P(\lambda)$.

Lemma 4.1 *(i)*

$$\Lambda_1 X_1^T M X_2 + X_1^T M X_2 \Lambda_2 + X_1^T D X_2 = 0;$$

(ii)

$$\Lambda_1 X_1^T M X_2 \Lambda_2 - X_1^T K X_2 = 0.$$

Proof:. We prove here only Part (i). The proof of Part (ii) is similar.

Proof of Part (i) : Since Λ_1 and Λ_2 are eigenvalue matrices of $P(\lambda)$ and X_1 and X_2 are the associate eigenvector matrices, we have

$$M X_1 \Lambda_1^2 + D X_1 \Lambda_1 + K X_1 = 0. \tag{6}$$

$$M X_2 \Lambda_2^2 + D X_2 \Lambda_2 + K X_2 = 0. \tag{7}$$

From (6), we get

$$-K X_1 = M X_1 \Lambda_1^2 + D X_1 \Lambda_1.$$

Taking the transpose of the both sides gives rise to

$$-X_1^T K = \Lambda_1^2 X_1^T M + \Lambda_1 X_1^T D.$$

Post-multiplying both sides of the above equation by X_2 we get

$$-X_1^T K X_2 = \Lambda_1^2 X_1^T M X_2 + \Lambda_1 X_1^T D X_2. \tag{8}$$

Also, pre-multiplying both sides of (7) by X_1^T we get

$$X_1^T M X_2 \Lambda_2^2 + X_1^T D X_2 \Lambda_2 + X_1^T K X_2 = 0. \tag{9}$$

From (8) and (9) we have

$$\Lambda_1^2 X_1^T M X_2 + \Lambda_1 X_1^T D X_2 = X_1^T M X_2 \Lambda_2^2 + X_1^T D X_2 \Lambda_2$$

or

$$\Lambda_1(\Lambda_1 X_1^T M X_2 + X_1^T D X_2) = (X_1^T M X_2 \Lambda_2 + X_1^T D X_2)\Lambda_2. \tag{10}$$

Adding the term $\Lambda_1 X_1^T M X_2 \Lambda_2$ to both sides of equation (10) gives

$$\Lambda_1(\Lambda_1 X_1^T M X_2 + X_1^T M X_2 \Lambda_2 + X_1^T D X_2) = (\Lambda_1 X_1^T M X_2 + X_1^T M X_2 \Lambda_2 + X_1^T D X_2)\Lambda_2. \tag{11}$$

Let $\emptyset = (\theta_{ij}) := \Lambda_1 X_1^T M X_2 + X_1^T M X_2 \Lambda_2 + X_1^T D X_2$. It then follows from (11) that

$$(\lambda_j - \lambda_{k+p})\,\theta_{jk} = 0 \quad \text{for } 1 \le j \le p \text{ and } 1 \le k \le 2n - p. \tag{12}$$

By the assumption that $\{\lambda_1, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_{2n}\} = \emptyset$, we have $\theta_{jk} = 0$ for all $1 \le j \le p$, $1 \le k \le 2n - p$. This completes the proof. □

Remark 4.2 *Lemma 4.1 can also be derived from an orthogonality relation proved earlier by Brahama and Datta [3, 4]. But there it was assumed that $\lambda_1, \ldots, \lambda_p$ are different from zero. No such assumption has been made here. This has a consequence that one does not have to be restricted to be assigning only nonzero eigenvalues. The zero open-loop eigenvalues can also be reassigned, if needed.*

Theorem 4.3 *Let $\Gamma = [\gamma_1, \ldots, \gamma_p] \in \mathbb{C}^{m \times p}$ be a nonzero matrix such that if $\mu_j = \bar{\mu}_k$, then $\gamma_j = \bar{\gamma}_k$.*

(i) For any arbitrary choice of Φ, the feedback matrices F_1 and F_2 defined by

$$F_1 = M X_1 \Phi^T$$

and

$$F_2 = (M X_1 \Lambda_1 + D X_1)\Phi^T$$

are such that

$$M X_2 \Lambda_2^2 + (D - B F_1^T) X_2 \Lambda_2 + (K - B F_2^T) X_2 = 0$$

(That is, there will be no spill-over).

(ii) If Φ is chosen satisfying the $p \times p$ linear system: $\Phi Z = \Gamma$, where Γ is arbitrary and Z is obtained by solving the Sylvester equation:

$$\Lambda_1 Z - Z \Lambda_1' = -X_1^T B \Gamma, \tag{13}$$

then F_1 and F_2 are real and the closed-loop eigenvalues will include $\mu_1,\ \mu_2, \ldots, \mu_p$ with the corresponding eigenvectors $y_1, y_2, \ldots, y_p$.

(iii) Suppose that Z is invertible. Then $S := [F_2^T, F_1^T] = \Gamma Z^{-1} C$ where $C = [\Lambda_1 X_1^T M + X_1^T D,\ X_1^T M]$.

Sketch of the Proof:

$A.$ Part (i) is proved by using the orthogonality relations given in Lemma 4.1.

$B.$ The proof of Part (ii) amounts to showing that

$$MY_1\Lambda_1'^2 + (D - BF_1^T)Y_1\Lambda_1' + (K - BF_2^T)Y_1 = 0,$$

which is equivalent to showing that

$$MY_1\Lambda_1'^2 + DY_1\Lambda_1' + KY_1 = B\Phi Z,$$

where

$$Z = \Lambda_1 X_1^T M Y_1 \Lambda_1' - X_1^T K Y_1.$$

Proof of Part (ii) now follows by setting $\Phi Z = \Gamma$ and noting that Z solves the Sylvester equation (13).

$C.$ Proof of Part (iii) is easy.

Details of the proof are omitted, because they are similar to the proof given in Brahma and Datta [4].

4.2 Expressing the matrix Y and Y^{-1} in terms of the Parametric Matrix Γ

To express Y in terms of Γ, we first note that Y can be expressed as (Brahama and Datta [4], Qian and Xu [33]):

$$Y = \begin{bmatrix} Y_1 & X_2 \\ Y_1\Lambda_1' & X_2\Lambda_2 \end{bmatrix}. \tag{14}$$

Let $Y_1 = (y_1, y_2, \ldots, y_p)$. Then each $y_k,\ k = 1, \ldots, p$ is uniquely determined by the system of equations:

$$(\mu_k^2 M + \mu_k D + K)\mathbf{y}_k = B\gamma_k. \tag{15}$$

Thus, Y_1 is a function of the parametric matrix Γ. Also, the (unknown) matrices X_2 and Λ_2 are fixed and do not take part in feedback computations, so they are independent of Γ.

We now show how to express Y^{-1} as a function of Γ. To do so, we first state the following preliminary lemma.

Lemma 4.4 [37, Lemma 1.3] *Let $A_1 \in \mathbb{C}^{p\times m}$, $A_2 \in \mathbb{C}^{n\times q}$, $A_3 \in \mathbb{C}^{p\times q}$, $E \in \mathbb{C}^{m\times n}$ be given. Define*

$$\Omega := \{E \in \mathbb{C}^{m\times n} : A_1 E A_2 = A_3\}.$$

Then $\Omega \neq \emptyset$ if and only if A_1, A_2, A_3 satisfy

$$A_1 A_1^+ A_3 A_2^+ A_2 = A_3,$$

and in case of $\Omega \neq \emptyset$, *any* $E \in \Omega$ *can be expressed as:*

$$E = A_1^+ A_3 A_2^+ + T - A_1^+ A_1 T A_2 A_2^+,$$

where $T \in \mathbb{C}^{m\times n}$. *Moreover, there is a unique matrix* $E^{(0)} \in \Omega$ *given by*

$$E^{(0)} = A_1^+ A_3 A_2^+ + E^* - A_1^+ A_1 E^* A_2 A_2^+$$

such that for any unitarily invariant norm $\|\cdot\|$,

$$\|E^{(0)} - E^*\| = \min_{E\in\Omega} \|E - E^*\|.$$

The next theorem shows how the inverse of the nonsingular matrix Y is determined.

Theorem 4.5 *Suppose that the nontrivial matrix* $\Gamma = \{\gamma_1, \dots, \gamma_p\} \in \mathbb{C}^{m\times p}$ *is such that if* $\mu_j = \bar{\mu}_k$, *then* $\gamma_j = \bar{\gamma}_k$. *Let* Z *satisfy (13) and suppose that* Z *is invertible. Also, let the matrices* $F_1 = MX_1\Phi^T$, $F_2 = (MX_1\Lambda_1 + D\Lambda_1)\Phi^T$, *where* Φ *is determined by* $\Phi Z = \Gamma$, *and let the matrices* S *and* C *be defined as in Theorem 4.3 and the* $2n$*-by-*$2n$ *complex matrix* Y *defined in (14) be nonsingular. Let*

$$Y := [\tilde{Y}_1,\, \tilde{X}_2], \quad \tilde{Y}_1 \in \mathbb{C}^{2n\times p}, \quad \tilde{X}_2 \in \mathbb{C}^{2n\times(2n-p)}.$$

Then the PQEVAP is solvable if Y^{-1} *is given by*

$$Y^{-1} = \begin{bmatrix} Z^{-1}C \\ \tilde{X}_2^+(I - \tilde{Y}_1 Z^{-1}C) \end{bmatrix}. \tag{16}$$

Proof: It easily follows that the PQVEAP is solvable if and only if

$$M[Y_1\Lambda_1'^2, X_2\Lambda_2^2] + (D - BF_1^T)[Y_1\Lambda_1', X_2\Lambda_2] + (K - BF_2^T)[Y_1, X_2] = 0.$$

or

$$BF_1^T[Y_1\Lambda_1', X_2\Lambda_2] + BF_2^T[Y_1, X_2] = M[Y_1\Lambda_1'^2, X_2\Lambda_2^2] + D[Y_1\Lambda_1', X_2\Lambda_2] + K[Y_1, X_2]. \tag{17}$$

It is easy to know that (17) holds if and only if Y satisfies

$$BSY = [MY_1\Lambda_1'^2 + DY_1\Lambda_1' + KY_1,\, MX_2\Lambda_2^2 + DX_2\Lambda_2 + KX_2]. \tag{18}$$

By (15), we have

$$MY_1\Lambda_1'^2 + DY_1\Lambda_1' + KY_1 = B\Gamma. \tag{19}$$

Using the notations X_2 and Λ_2, we get

$$MX_2\Lambda_2^2 + DX_2\Lambda_2 + KX_2 = 0. \tag{20}$$

Substituting (19) and (20) into (18) gives rise to

$$BSY = [B\Gamma, 0] = B[\Gamma, 0].$$

Since the matrix B is full column rank and the eigenvector matrix Y is nonsingular, we have

$$SY = [\Gamma, 0]. \tag{21}$$

By (iii) of Theorem 4.3, we know that $S = \Gamma Z^{-1} C$. Then (21) is equivalent to

$$\Gamma Z^{-1} C = \Gamma [I_p, 0] Y^{-1}. \tag{22}$$

It is observed that (22) holds if

$$[I_p, 0] Y^{-1} = Z^{-1} C. \tag{23}$$

By Lemma 4.4, the equation $[I_p, 0]E = Z^{-1}C$ has a solution as follows:

$$E_Y^{(0)} = [I_p, 0]^+ Z^{-1} C + (I - [I_p, 0]^+ [I_p, 0]) Y^{-1}, \tag{24}$$

which is closest to Y^{-1} in the sense of any unitarily invariant norm. By using (23), it is easy to check that $E_Y^{(0)} Y = I$. Thus, $Y^{-1} = E_Y^{(0)}$. Let

$$Y^{-1} := \begin{bmatrix} W_1 \\ W_2 \end{bmatrix}, \quad W_1 \in \mathbb{R}^{p \times 2n}, \quad W_2 \in \mathbb{R}^{(2n-p) \times 2n}.$$

Then, by (24), we obtain that

$$W_1 = Z^{-1} C. \tag{25}$$

Next, we compute the expression of W_2. By $YY^{-1} = I$, we have

$$\tilde{X}_2 W_2 = I - \tilde{Y}_1 W_1. \tag{26}$$

Let the SVD (cf. [22]) of the matrix $\tilde{X}_2$ be given by

$$\begin{array}{l} \tilde{X}_2 = \tilde{U} \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} \tilde{V}, \quad \tilde{U} = [\tilde{U}_1, \tilde{U}_2], \quad \tilde{U}_1 \in \mathbb{C}^{2n \times (2n-p)}, \\ \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_{2n-p}), \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{2n-p} > 0. \end{array}$$

Premultiplying (26) by $\tilde{U}^H$ gives

$$\begin{cases} \Sigma \tilde{V}^H W_2 &= \tilde{U}_1^H (I - \tilde{Y}_1 W_1), \\ 0 &= \tilde{U}_2^H (I - \tilde{Y}_1 W_1). \end{cases}$$

Thus,

$$W_2 = \tilde{V} \Sigma^{-1} \tilde{U}_1^H (I - \tilde{Y}_1 W_1).$$

This, together with (25) leads to (16). □

Expressing the Cost Function J as a Function of Γ:

In view of Theorem 4.5 and using (25) and (26), we can further reduce the cost function J as follows:

$$\begin{aligned} J \; = \; & \tfrac{1}{2}\alpha(\|Y_1\|_F^2 + \|Y_1 \Lambda_1'\|_F^2 + \|W_1\|_F^2 + \|W_2\|_F^2) \\ & + \tfrac{1}{2}\alpha(\|X_2\|_F^2 + \|X_2 \Lambda_2\|_F^2) + \tfrac{1}{2}(1-\alpha)(\|F_1\|_F^2 + \|F_2\|_F^2). \end{aligned} \tag{27}$$

We note that Λ_2 is a fixed matrix, and X_2 is independent of Γ. Also, from Theorem 4.3 and (15), we see that the matrices F_1, F_2, Y_1, W_1, and W_2 are the functions of Γ.

The expression (27) still contains the term W_2 which involves $\tilde{X}_2^+$. However, we note that

$$\|W_2\|_F \leq \|\tilde{X}_2^+\|_2\|(I - \tilde{Y}_1 W_1)\|_F = 1/\sigma_{\min}(\tilde{X}_2)\,\|I - \tilde{Y}_1 W_1\|_F,$$

where $\sigma_{\min}(\tilde{X}_2)$ denotes the smallest singular value of $\tilde{X}_2$. By assumption that $\tilde{X}_2$ has linearly independent columns, so we may expect that $\sigma_{\min}(\tilde{X}_2)$ is not too small. So, instead of minimizing J given by (27), we will minimize the following expression, which also for simplicity, we write as J:

$$J = \alpha J_1 + (1-\alpha)J_2 + \alpha J_3, \tag{28}$$

where

$$\begin{cases} J_1 &= \frac{1}{2}(\|Y_1\|_F^2 + \|Y_1\Lambda_1'\|_F^2 + \|W_1\|_F^2 + \beta\|I - \tilde{Y}_1 W_1\|_F^2), \\ J_2 &= \frac{1}{2}(\|F_1\|_F^2 + \|F_2\|_F^2), \\ J_3 &= \frac{1}{2}(\|X_2\|_F^2 + \|X_2\Lambda_2\|_F^2) \end{cases}$$

with the parameter $\beta > 0$ being a-priori estimate for the upper bound of $1/\sigma_{\min}^2(\tilde{X}_2)$.

Since J_3 is independent of Γ, we can now focus on the following unconstrained minimization problem:

$$\begin{aligned} \min_{\Gamma} \alpha J_1 + (1-\alpha)J_2 &= \tfrac{1}{2}\alpha(\|Y_1\|_F^2 + \|Y_1\Lambda_1'\|_F^2 + \|W_1\|_F^2 + \beta\|I - \tilde{Y}_1 W_1\|_F^2) \\ &+ \tfrac{1}{2}(1-\alpha)(\|F_1\|_F^2 + \|F_2\|_F^2). \end{aligned} \tag{29}$$

4.3 Gradient Expressions in Terms of Λ_1, X_1 and Y_1

The next theorem gives the gradient formula of J_2 in term of the known quantities Λ_1, Λ_1', X_1, and B.

Theorem 4.6 *Let $\Gamma = [\gamma_1, \ldots, \gamma_p] \in \mathbb{C}^{m\times p}$ be such that if $\mu_j = \bar{\mu}_k$, then $\gamma_j = \bar{\gamma}_k$. Let Z be the unique solution of the Sylvester equation:*

$$\Lambda_1 Z - Z\Lambda_1' = -X_1^T B\Gamma.$$

Let $F_1 = MX_1\Phi^T$ and $F_2 = (MX_1\Lambda_1 + DX_1)\Phi^T$, where Φ is determined by $\Phi Z = \Gamma$. Suppose that Z is nonsingular and Υ satisfies the Sylvester equation:

$$\Lambda_1'\Upsilon - \Upsilon\Lambda_1 = -Z^{-1}CS^H\Phi.$$

Then

$$\nabla_\Gamma J_2 = \frac{1}{2}[Z^{-1}CS^H - \Upsilon X_1^T B]^T.$$

Proof: The result can be proved similarly as in Theorem 2 in [4]. We omit the details here. □

In the following, we shall provide an explicit formula of the gradient of J_1 with respect to the parameter Γ via the Sylvester equation-based parameterization. By Theorem 4.3, we prove the following result on the gradient of J_1 with respect to Γ.

Theorem 4.7 *Let the nontrivial matrix $\Gamma = \{\gamma_1, \dots, \gamma_p\} \in \mathbb{C}^{m\times p}$ be such that if $\mu_j = \bar{\mu}_k$, then $\gamma_j = \bar{\gamma}_k$. Let Z be the unique solution of the Sylvester equation (13). Let $F_1 = MX_1\Phi^T$ and $F_2 = (MX_1\Lambda_1 + DX_1)\Phi^T$, where Φ is determined by $\Phi Z = \Gamma$. Define the matrices W_1 and W_2 by (25) and (26). Suppose that Z is invertible and Y_1, U, U_1, U_2, V, V_1, and V_2, respectively, satisfy the following equations:*

$$MY_1\Lambda_1'^2 + DY_1\Lambda_1' + KY_1 = B\Gamma, \tag{30}$$

$$\begin{cases} MU\Lambda_1'^2 + DU\Lambda_1' + KU = [(I + \Lambda_1'\bar{\Lambda}_1')Y_1^H]^T, \\ MU_1\Lambda_1'^2 + DU_1\Lambda_1' + KU_1 = [W_1W_{11}^H]^T, \\ MU_2\Lambda_1'^2 + DU_2\Lambda_1' + KU_2 = [\Lambda_1'W_1W_{21}^H]^T \end{cases} \tag{31}$$

and

$$\begin{cases} \Lambda_1'V - V\Lambda_1 = -W_1W_1^HZ^{-1}, \\ \Lambda_1'V_1 - V_1\Lambda_1 = -W_1W_{11}^HY_1Z^{-1}, \\ \Lambda_1'V_2 - V_2\Lambda_1 = -W_1W_{21}^HY_1\Lambda_1'Z^{-1}, \end{cases} \tag{32}$$

where

$$W_{11} = E_1 - Y_1W_1 \quad \text{and} \quad W_{21} = E_2 - Y_1\Lambda_1'W_1 \tag{33}$$

with $E_1 := [I_n, 0] \in \mathbb{R}^{n\times 2n}$ and $E_2 := [0, I_n] \in \mathbb{R}^{n\times 2n}$. Then, the gradient $\nabla_\Gamma J_1$ of J_1 with respect to Γ is given by

$$\begin{aligned} \nabla_\Gamma J_1 &= \tfrac{1}{2}[U^TB - VX_1^TB]^T \\ &- \tfrac{1}{2}\beta[(U_1+U_2)^TB - (V_1+V_2)X_1^TB]^T. \end{aligned} \tag{34}$$

Proof: We rewrite the function J_1 defined in (29) in the form:

$$J_1 = J_{11} + J_{12} + \beta J_{13}, \tag{35}$$

where $J_{11} = \frac{1}{2}\|Y_1\|_F^2 + \frac{1}{2}\|Y_1\Lambda_1'\|_F^2$ and $J_{12} = \frac{1}{2}\|W_1\|_F^2$, and $J_{13} = \|I - \tilde{Y}_1W_1\|_F^2$. It follows that

$$\nabla_\Gamma J_1 = \nabla_\Gamma J_{11} + \nabla_\Gamma J_{12} + \beta\nabla_\Gamma J_{13}. \tag{36}$$

We first give the formula for the gradient $\nabla_\Gamma J_{11}$ of J_{11}. For $J_{11} = \frac{1}{2}\mathrm{tr}([I + \Lambda_1'\bar{\Lambda}_1']Y_1^HY_1)$, we have the gradient $\nabla_\Gamma J_{11}$ from the first order variation as:

$$\begin{aligned} \Delta J_{11} &= \tfrac{1}{2}\mathrm{tr}([I + \Lambda_1'\bar{\Lambda}_1']Y_1^H\Delta Y_1) + \tfrac{1}{2}\mathrm{tr}([I + \Lambda_1'\bar{\Lambda}_1']\Delta Y_1^HY_1) \\ &= \tfrac{1}{2}\mathrm{tr}([I + \Lambda_1'\bar{\Lambda}_1']Y_1^H\Delta Y_1) + \tfrac{1}{2}\mathrm{tr}(Y_1[I + \Lambda_1'\bar{\Lambda}_1']\Delta Y_1^H) \\ &= \tfrac{1}{2}\mathrm{tr}(Z_1\Delta Y_1) + \tfrac{1}{2}\mathrm{tr}(Z_1^H\Delta Y_1^H), \end{aligned} \tag{37}$$

where $Z_1 = [I + \Lambda_1'\bar{\Lambda}_1']Y_1^H$.
We now express $\mathrm{tr}(Z_1\Delta Y_1)$ in terms of $\Delta\Gamma$. From (15), we know that ΔY_1 is uniquely determined by

$$M\Delta Y_1\Lambda_1'^2 + D\Delta Y_1\Lambda_1' + K\Delta Y_1 = B\Delta\Gamma \tag{38}$$

or

$$\mathrm{vec}(\Delta Y_1) = Z_2^{-1} Z_3 \mathrm{vec}(\Delta\Gamma),$$

where $Z_2 = \Lambda_1'^2 \otimes M + \Lambda_1' \otimes D + I_p \otimes K$ and $Z_3 = I_p \otimes B$. From the fact that if $A_1 \in \mathbb{C}^{m\times n}$ and $A_2 \in \mathbb{C}^{n\times m}$, then $\mathrm{tr}(A_1 A_2) = \mathrm{vec}(A_1^T)^T \mathrm{vec}(A_2)$, we get

$$\begin{aligned} \mathrm{tr}(Z_1 \Delta Y_1) &= \mathrm{vec}(Z_1^T)^T \mathrm{vec}(\Delta Y_1) \\ &= \mathrm{vec}(Z_1^T)^T Z_2^{-1} Z_3 \mathrm{vec}(\Delta\Gamma) \\ &= \left(Z_2^{-T} \mathrm{vec}(Z_1^T)\right)^T \mathrm{vec}(B\Delta\Gamma) \\ &= \mathrm{tr}(U^T B \Delta\Gamma), \end{aligned} \tag{39}$$

where U is determined by (31).
Next, we derive an expression for the term $\mathrm{tr}(Z_1^H \Delta Y_1^H)$ in (37). Taking the conjugate transpose of (38) gives

$$\bar{\Lambda}_1'^2 \Delta Y_1^H M + \bar{\Lambda}_1' \Delta Y_1^H D + \Delta Y_1^H K = \Delta\Gamma^H B^T$$

or

$$\mathrm{vec}(\Delta Y_1^H) = Z_4^{-1} Z_5 \mathrm{vec}(\Delta\Gamma^H),$$

where $Z_4 = M \otimes \bar{\Lambda}_1'^2 + D \otimes \bar{\Lambda}_1' + K \otimes I_p$ and $Z_5 = B \otimes I_p$. Then we have

$$\begin{aligned} \mathrm{tr}(Z_1^H \Delta Y_1^H) &= \mathrm{vec}(\bar{Z}_1)^T \mathrm{vec}(\Delta Y_1^H) \\ &= \mathrm{vec}(\bar{Z}_1)^T Z_4^{-1} Z_5 \mathrm{vec}(\Delta\Gamma^H) \\ &= \left(Z_4^{-T} \mathrm{vec}(\bar{Z}_1)\right)^T \mathrm{vec}(\Delta\Gamma^H B^T) \\ &= \mathrm{tr}(B^T \bar{U} \Delta\Gamma^H). \end{aligned} \tag{40}$$

Substituting (39) and (40) into (37) gives rise to

$$\Delta J_{11} = \frac{1}{2}\mathrm{tr}(U^T B \Delta\Gamma) + \frac{1}{2}\mathrm{tr}(B^T \bar{U} \Delta\Gamma^H),$$

and from which the gradient of J_{11} is given by

$$\nabla_\Gamma J_{11} = \frac{1}{2}[U^T B]^T. \tag{41}$$

Next, we establish the gradient $\nabla_\Gamma J_{12}$. For $J_{12} = \frac{1}{2}\mathrm{tr}(W_1^H W_1)$, we have

$$\begin{aligned} \Delta J_{12} &= \tfrac{1}{2}\mathrm{tr}(W_1^H \Delta W_1) + \tfrac{1}{2}\mathrm{tr}(\Delta W_1^H W_1) \\ &= \tfrac{1}{2}\mathrm{tr}(W_1^H \Delta W_1) + \tfrac{1}{2}\mathrm{tr}(W_1 \Delta W_1^H). \end{aligned} \tag{42}$$

We first deduce the expression of $\mathrm{tr}(W_1^H \Delta W_1)$. By (25), we get

$$\Delta W_1 = -Z^{-1} \Delta Z Z^{-1} C = -Z^{-1} \Delta Z W_1. \tag{43}$$

Then

$$\mathrm{tr}(W_1^H \Delta W_1) = -\mathrm{tr}(W_1^H Z^{-1} \Delta Z W_1) = -\mathrm{tr}(W_1 W_1^H Z^{-1} \Delta Z). \tag{44}$$

Now, we express $\mathrm{tr}(W_1 W_1^H Z^{-1} \Delta Z)$ in terms of $\Delta\Gamma$. First, it follows from (13) that

$$\Lambda_1 \Delta Z - \Delta Z \Lambda_1' = -X_1^T B \Delta\Gamma. \tag{45}$$

It is well-known that ΔZ of the Sylvester equation can be written as [36]

$$\Delta Z = \sum_{j=0}^{p-1} \sum_{k=0}^{p-1} \phi_{jk} (\Lambda_1)^j (-X_1^T B \Delta\Gamma)(\Lambda_1')^k,$$

where ϕ_{jk} are scalars.
Let $Z_6 = W_1 W_1^H Z^{-1}$ and $Z_7 = X_1^T B \Delta\Gamma$. Then we have

$$\begin{aligned} \mathrm{tr}\,(Z_6 \Delta Z) &= \mathrm{tr}\left(\sum_{j=0}^{p-1} \sum_{k=0}^{p-1} Z_6 \phi_{jk} (\Lambda_1)^j (-Z_7)(\Lambda_1')^k \right) \\ &= \mathrm{tr}\left(\sum_{j=0}^{p-1} \sum_{k=0}^{p-1} \phi_{jk} (\Lambda_1')^k (-Z_6)(\Lambda_1)^j Z_7 \right) \\ &= \mathrm{tr}(V X_1^T B \Delta\Gamma), \end{aligned} \tag{46}$$

where V is the unique solution to (32). Substituting (46) into (44) gives rise to

$$\mathrm{tr}(W_1^H \Delta W_1) = -\mathrm{tr}(V X_1^T B \Delta\Gamma). \tag{47}$$

Next, we deduce an expression for the term $\mathrm{tr}(W_1 \Delta W_1^H)$ in (42). By (43), we have

$$\mathrm{tr}(W_1 \Delta W_1^H) = -\mathrm{tr}(W_1 W_1^H \Delta Z^H Z^{-H}) = -\mathrm{tr}(Z^{-H} W_1 W_1^H \Delta Z^H). \tag{48}$$

Similar to the proof of (47), we can show that

$$\mathrm{tr}(W_1 \Delta W_1^H) = -\mathrm{tr}\big([V X_1^T B]^H \Delta\Gamma^H\big). \tag{49}$$

It follows from (42), (47), and (49) that

$$\Delta J_{12} = -\frac{1}{2}\mathrm{tr}(V X_1^T B \Delta\Gamma) - \frac{1}{2}\mathrm{tr}\big([V X_1^T B]^H \Delta\Gamma^H\big). \tag{50}$$

Thus the gradient $\nabla_\Gamma J_{12}$ of J_{12} is given by

$$\nabla_\Gamma J_{12} = -\frac{1}{2}[V X_1^T B]^T. \tag{51}$$

Finally, we derive the expression of $\nabla_\Gamma J_{13}$ for J_{13}. By (33), we get

$$J_{13} = \frac{1}{2}\|I - \tilde{Y}_1 W_1\|_F^2 = \frac{1}{2}\mathrm{tr}(W_{11}^H W_{11}) + \frac{1}{2}\mathrm{tr}(W_{21}^H W_{21}).$$

Thus,

$$\Delta J_{13} = \frac{1}{2}\text{tr}(W_{11}^H \Delta W_{11}) + \frac{1}{2}\text{tr}(W_{11}\Delta W_{11}^H) + \frac{1}{2}\text{tr}(W_{21}^H \Delta W_{21}) + \frac{1}{2}\text{tr}(W_{21}\Delta W_{21}^H). \quad (52)$$

We first give an expression for $\text{tr}(W_{11}^H \Delta W_{11})$. By (33) and (43), we have

$$\Delta W_{11} = -\Delta Y_1 W_1 - Y_1 \Delta W_1 = -\Delta Y_1 W_1 + Y_1 Z^{-1} \Delta Z W_1.$$

Thus

$$\text{tr}(W_{11}^H \Delta W_{11}) = -\text{tr}(W_1 W_{11}^H \Delta Y_1) + \text{tr}(W_1 W_{11}^H Y_1 Z^{-1} \Delta Z). \quad (53)$$

We now express $\text{tr}(W_1 W_{11}^H \Delta Y_1)$ in terms of $\Delta\Gamma$. Similarly, for $\text{tr}(Z_1 \Delta Y_1)$ in (39), we can show that

$$\text{tr}(W_1 W_{11}^H \Delta Y_1) = \text{tr}(U_1^T B \Delta\Gamma), \quad (54)$$

where U_1 is determined by (31). By following the similar proof lines of (46), we obtain

$$\text{tr}(W_1 W_{11}^H Y_1 Z^{-1} \Delta Z) = \text{tr}(V_1 X_1^T B \Delta\Gamma), \quad (55)$$

where V_1 is determined by (32). Substituting (54) and (55) into (53) gives

$$\text{tr}(W_{11}^H \Delta W_{11}) = -\text{tr}([U_1^T B - V_1 X_1^T B]\Delta\Gamma). \quad (56)$$

Using a proof similar to (56), we can show that

$$\text{tr}(W_{11} \Delta W_{11}^H) = -\text{tr}([U_1^T B - V_1 X_1^T B]^H \Delta\Gamma^H). \quad (57)$$

Next, we express $\frac{1}{2}\text{tr}(W_{21}^H \Delta W_{21})$ in terms of $\Delta\Gamma$. By (33) and (43), we have

$$\Delta W_{21} = -\Delta Y_1 \Lambda_1' W_1 - Y_1 \Lambda_1' \Delta W_1 = -\Delta Y_1 \Lambda_1' W_1 + Y_1 \Lambda_1' Z^{-1} \Delta Z W_1.$$

Therefore,

$$\text{tr}(W_{21}^H \Delta W_{21}) = -\text{tr}(\Lambda_1' W_1 W_{21}^H \Delta Y_1) + \text{tr}(W_1 W_{21}^H Y_1 \Lambda_1' Z^{-1} \Delta Z). \quad (58)$$

We now express $\text{tr}(\Lambda_1' W_1 W_{21}^H \Delta Y_1)$ in terms of $\Delta\Gamma$. By the similar proof for $\text{tr}(Z_1 \Delta Y_1)$ in (39), we can show that

$$\text{tr}(\Lambda_1' W_1 W_{21}^H \Delta Y_1) = \text{tr}(U_2^T B \Delta\Gamma), \quad (59)$$

where U_2 is determined by (31). By following the similar proof lines of (46), we obtain

$$\text{tr}(W_1 W_{21}^H Y_1 \Lambda_1' Z^{-1} \Delta Z) = \text{tr}(V_2 X_1^T B \Delta\Gamma), \quad (60)$$

where V_2 is determined by (32). Substituting (59) and (60) into (58) gives

$$\text{tr}(W_{21}^H \Delta W_{21}) = -\text{tr}([U_2^T B - V_2 X_1^T B]\Delta\Gamma). \quad (61)$$

As with proof of (61), we can show that

$$\text{tr}(W_{21} \Delta W_{21}^H) = -\text{tr}([U_2^T B - V_2 X_1^T B]^H \Delta\Gamma^H). \quad (62)$$

Substituting (56), (57), (61), and (62) into (52), we get

$$\begin{aligned}\Delta J_{13} &= -\tfrac{1}{2}\mathrm{tr}([(U_1+U_2)^T B-(V_1+V_2)X_1^T B]\Delta\Gamma) \\ &\quad -\tfrac{1}{2}\mathrm{tr}([(U_1+U_2)^T B-(V_1+V_2)X_1^T B]^H \Delta\Gamma^H).\end{aligned} \tag{63}$$

Thus the gradient $\nabla_\Gamma J_{13}$ of J_{13} is given by

$$\nabla_\Gamma J_{13} = -\frac{1}{2}[(U_1+U_2)^T B-(V_1+V_2)X_1^T B]^T. \tag{64}$$

The proof of the required gradient formula now follows from (41), (51) and (64) and (36).

□

4.4 An Optimization Algorithm for Simultaneous Feedback Norm and Closed-Loop Condition Minimization

Based on above analysis, we write down the following algorithm for simultaneously minimizing feedback norms and the closed-loop condition number.

Algorithm 4.1:
Inputs:

1. The matrices $M, D, K \in \mathbb{R}^{n\times n}$, where $M^T = M > 0$, $K^T = K$, and $D^T = D$.
2. The control matrix $B \in \mathbb{R}^{n\times n}$ $(m \le n)$.
3. A self-conjugate subset $\{\lambda_1, \ldots, \lambda_p\}$ of the spectrum of $P(\lambda)$ and the corresponding eigenvectors.
4. A suitably chosen self-conjugate subset $\{\mu_1, \ldots, \mu_p\}$.
5. $\alpha \in [0,1]$ and $\beta \ge 0$.
6. $\epsilon =$ Tolerance limit for gradient.
7. $Max_{iter} =$ Maximum number of iterations.

Outputs:

The real feedback matrices F_1 and F_2 such that the spectrum of the close-loop pencil $P_c(\lambda)$ is the set $\{\mu_1, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n}\}$ and the objective function $\alpha J_1 + (1-\alpha)J_2$ defined in (29) is minimized.

Step 1. Form the matrices Λ_1, Λ_1', X_1, from the given eigenvalues and eigenvectors, and the matrix C defined in Theorem 4.3. Set $k = 1$.

Step 2. Choose a matrix $\Gamma = [\gamma_1, \ldots, \gamma_p] \in \mathbb{C}^{m\times p}$ $(m \le p)$ such that if $\bar{\mu}_j = \mu_k$, then $\bar{\gamma_j} = \gamma_k$.

Step 3. Compute the solution Z of the Sylvester equation $\Lambda_1 Z - Z\Lambda_1' = -X_1^T B\Gamma$. (This requires $O(npm + p^3)$ flops. It is done by the MATLAB function `lyap`). If the condition number of Z is large, choose another Γ.

Step 4. Solve the linear system $\Phi Z = \Gamma$ for Φ. (This requires $O(p^3)$ operations).

Step 5. Form the matrix $S = \Phi C$. (This requires $O(nmp)$ flops).

Step 6. Solve the equation $MY_1\Lambda_1'^2 + DY_1\Lambda_1' + KY_1 = B\Gamma$ for Y_1. (This requires $O(n^3 p)$ operations).

Step 7. Solve for the matrices U, U_1, and U_2 as follows:

7.1 Solve for U: $MU\Lambda_1'^2 + DU\Lambda_1' + KU = ((I + \Lambda_1'\bar{\Lambda}_1')Y_1^H]^T$

7.2 Solve for U_1: $MU_1\Lambda_1'^2 + DU_1\Lambda_1' + KU_1 = [W_1 W_{11}^H]^T$

7.3 Solve for U_2: $MU_2\Lambda_1'^2 + DU_2\Lambda_1' + KU_2 = [\Lambda_1' W_1 W_{21}^H]^T$

(These computations require $O(n^3 p)$ flops).

Step 8. Compute Υ, V, V_1, and V_2 by solving the following Sylvester equations:

8.1 Solve for Υ: $\Lambda_1'\Upsilon - \Upsilon\Lambda_1 = -Z^{-1}CS^H\Phi$,

8.2 Solve for V: $\Lambda_1' V - V\Lambda_1 = -W_1 W_1^H Z^{-1}$,

8.3 Solve for V_1: $\Lambda_1' V_1 - V_1\Lambda_1 = -W_1 W_{11}^H Y_1 Z^{-1}$,

8.4 Solve for V_2: $\Lambda_1' V_2 - V_2\Lambda_1 = -W_1 W_{21}^H Y_1 \Lambda_1' Z^{-1}$

(These steps require $O(p^3 + np^2)$ flops).

Step 9. Compute the gradient $Grad := \alpha\nabla_\Gamma J_1 + (1-\alpha)\nabla_\Gamma J_2$, using results from Theorems 4.7 and 4.6, respectively.

(This requires $O(p^3 + np^2)$ operations). If $\|Grad\|_F \leq \epsilon$ or if the number of iterations exceeds Max_{iter}, go to Step 6; Otherwise, go to Step 10.

Step 10. Compute a new Γ (as shown below) using a gradient-based optimization method (we use the Broyden-Fletcher-Goldfarb-Shanno **(BFGS)** method, which is described in details in [31, Chap.8]). Set $k = k + 1$ and return to step 2.

Step 11. For the Γ which gives the minimum value of $\alpha J_1 + (1-\alpha)J_2$, compute the matrices F_1 and F_2 as in Theorem 4.3. Stop.

Computation of New Γ in Step 10:

The function to be minimized is $J(\Gamma) = \frac{1}{2}\alpha(\|Y(\Gamma)\|_F^2 + \|Y^{-1}(\Gamma)\|_F^2) + \frac{1}{2}(1-\alpha)(\|F_1(\Gamma)\|_F^2 + \|F_2(\Gamma)\|_F^2)$. We denote the current value of Γ by Γ_{old} and the new value of Γ by Γ_{new}. Then Γ_{new} is obtained as follows:

a) Replace Γ_{old} by $\hat{\Gamma} = \Gamma_{old} + \delta d_j$, where d_j is the search direction which, in the BFGS method, is given by $d_j = -D_j Grad$. Here *Grad* represents the current value of the gradient and D_j is the metric obtained as in the BFGS method if $j > 1$ and equal to D_1 if $j = 1$.

b) Let $\hat{Z}$ be the solution of the Sylvester equation: $\Lambda_1 \hat{Z} - \hat{Z}\Lambda_1' = -X_1^T B \hat{\Gamma}$. Let $\hat{\Phi} = \hat{\Gamma}\hat{Z}^{-1}$ be the solution of the linear system: $\hat{\Phi}\hat{Z} = \hat{\Gamma}$. Compute the value $\hat{Y}_1$ of Y_1 corresponding to $\hat{\Gamma}$ as follows: Let $\hat{Y}_1 = [\hat{\mathbf{y}}_1, \ldots, \hat{\mathbf{y}}_p]$ and $\hat{\Gamma} = [\hat{\gamma}_1, \ldots, \hat{\gamma}_p]$. Then $\hat{\mathbf{y}}_k$ is uniquely determined by $(\mu_k^2 M + \mu_k D + K)\hat{\mathbf{y}}_k = \hat{\mathbf{y}}_k$ for $k = 1, \ldots, p$. Form $\hat{S} = \hat{\Phi}C$, $\hat{W}_1 = \hat{Z}^{-1}C$, $\hat{W}_{11} = E_1 - \hat{Y}_1\hat{W}_1$, and $\hat{W}_{21} = E_2 - \hat{Y}_1\Lambda_1'\hat{W}_1$.

c) Find $l = \min_\delta\{\frac{1}{2}\alpha(\|\hat{Y}_1\|_F^2 + \|\hat{Y}_1\Lambda_1'\|_F^2 + \|\hat{W}_1\|_F^2 + \beta\|\hat{W}_{11}\|_F^2 + \beta\|\hat{W}_{21}\|_F^2) + \frac{1}{2}(1-\alpha)\|\hat{S}\|_F^2\}$. This is done by using the MATLAB function `fminbnd`.

d) $\Gamma_{new} = \Gamma_{old} + l d_j$.

Remarks:

In order to implement the control law (3) with the feedback matrices computed by Algorithm 4.1, knowledge of this displacement vector $x(t)$ and the velocity vector $\dot{x}(t)$ is needed. To this end, an algorithm to estimate these quantities by means of an observer has been recently proposed by Carvalho and Datta [6]. Experiments are currently underway to implement the control law (3) on real-life structures by integrating this technique with our feedback formulas.

5 Numerical Experiments

In this section, we will state some numerical results for solving the RPQVEAP using Algorithm 4.1. The algorithm was implemented in `MATLAB 7.0` and run on a PC Intel Pentium IV of 3.00 GHZ CPU. In our computational experiments, we use the tolerance $\epsilon = 1.0 \times 10^{-6}$ and the maximum number of iterations, $Max_{iter} = 500$. The algorithm is applied to five different test problems.

Problem 5.1 *Here the three matrices M, D, K are randomly generated correlation matrices by* `MATLAB` *routine function* `gallery('randcorr',n)` *and B is random matrix, i.e.,*

$$M = \begin{bmatrix} 1.0000 & -0.1067 & -0.1895 & -0.5170 & 0.1696 \\ -0.1067 & 1.0000 & 0.3893 & 0.1415 & 0.2020 \\ -0.1895 & 0.3893 & 1.0000 & -0.1305 & 0.2989 \\ -0.5170 & 0.1415 & -0.1305 & 1.0000 & 0.1928 \\ 0.1696 & 0.2020 & 0.2989 & 0.1928 & 1.0000 \end{bmatrix},$$

$$D = \begin{bmatrix} 1.0000 & -0.3784 & 0.2015 & 0.0388 & 0.1476 \\ -0.3784 & 1.0000 & -0.0072 & -0.1233 & 0.2483 \\ 0.2015 & -0.0072 & 1.0000 & 0.3487 & -0.2794 \\ 0.0388 & -0.1233 & 0.3487 & 1.0000 & 0.1081 \\ 0.1476 & 0.2483 & -0.2794 & 0.1081 & 1.0000 \end{bmatrix},$$

$$K = \begin{bmatrix} 1.0000 & 0.6591 & -0.4258 & 0.1630 & 0.6081 \\ 0.6591 & 1.0000 & -0.4137 & 0.0308 & 0.0804 \\ -0.4258 & -0.4137 & 1.0000 & -0.4927 & -0.2494 \\ 0.1630 & 0.0308 & -0.4927 & 1.0000 & 0.4947 \\ 0.6081 & 0.0804 & -0.2494 & 0.4947 & 1.0000 \end{bmatrix}, \quad B = \begin{bmatrix} 0.3357 & 0.8105 \\ 0.8484 & 0.9999 \\ 0.2499 & 0.8417 \\ 0.0228 & 0.7662 \\ 0.3405 & 0.8373 \end{bmatrix}.$$

The open-loop eigenvalues are: $\{-6.4634, -0.3743 \pm 1.6463i, -0.4769 \pm 1.0401i, -0.2714 \pm 0.7579i, -0.0855, -0.3804 \pm 0.2495i\}$*, where* $i := \sqrt{-1}$*. The eigenvalue* $\{-0.0855\}$ *was reassigned to* $\{-0.1\}$ *and the other eigenvalues were kept unchanged.*

Problem 5.2 *See [30]. The matrices* $M, D, K,$ *and* B *here are given by*

$$M = 10I_3, \quad D = 0, \quad K = \begin{bmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{bmatrix}.$$

The open-loop eigenvalues are: $\{\pm 3.6039i, \pm 2.4940i, \pm 0.8901i\}$*. In this example, the first two eigenvalues* $\{\pm 3.6039i\}$ *were reassigned to* $\{-1, -2\}$*, the other eigenvalues were kept unchanged.*

Problem 5.3 *The matrices* $M, D, K,$ *and* B *here are taken as follows:*

$$M = I_3, \quad D = \begin{bmatrix} 2.5 & 2.0 & 0 \\ 2.0 & 1.7 & 0.4 \\ 0 & 0.4 & 2.5 \end{bmatrix}, \quad K = \begin{bmatrix} 16 & 12 & 0 \\ 12 & 13 & 4 \\ 0 & 4 & 29 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{bmatrix}.$$

The open-loop eigenvalues are: $\{-0.0129 \pm 1.4389i, -1.3342 \pm 5.2311i, -2.0030 \pm 4.7437i\}$*. We change the first two open-loop eigenvalues:* $\{-0.0129 \pm 1.4389i\}$ *to* $\{-0.1 \pm 1.4389i\}$*. The other eigenvalues are kept unchanged.*

Problem 5.4 *See [4, Problem 1:] In this example,* $M, D, K,$ *and* B *are taken as follows:*

$$M = 4I_n, \quad D = 4I_n \quad K = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

We take $n = 10$*. Then there are* 20 *open-loop eigenvalues. The sixteenth and the seventeenth eigenvalues* $\{0.0000,\ -0.0251\}$ *were reassigned to* $\{-0.1,\ -0.2\}$*, and the other eigenvalues were kept unchanged.*

Performance of Algorithm 4.1

We display in the following table the results of Algorithm 4.1 with three different values of α, $\alpha = 0, 0.5, 1$; and with three different values of β, $\beta = 1, 10, 100$. Note that

- $\alpha = 0$ corresponds to the minimum feedback norm problem and of interests there are the quantities $\|F_1\|_F$ and $\|F_2\|_F$.
- $\alpha = 1$ corresponds to RPQPEVAP and of interest there is $\text{cond}_F(Y)$.

- $\alpha = 0.5$, corresponds to simultaneous minimization of feedback norms and the sensibility of the closed-loop eigenvalues with equal weights. The quantities of interests here are both $\|F\|_F, \|F\|_2$, and $\text{cond}_p(Y)$.

Interpretation: The results in Table 1 below, show that the proposed algorithm worked well in all cases:

For $\alpha = 0$, $\|F\|_F$ and $\|F_2\|_F$ are small, for $\alpha = 1$, $\text{cond}_F(Y)$ is small and for $\alpha = 0.5$, $\|F_1\|_F$, $\|F_2\|_F$ and $\text{cond}_F(Y)$ are all small.

$\beta = 1$									
	$\alpha = 1$			$\alpha = 0.5$			$\alpha = 0$		
P.	$\|F_1\|_F$	$\|F_2\|_F$	$\kappa_2(Y)$	$\|F_1\|_F$	$\|F_2\|_F$	$\kappa_2(Y)$	$\|F_1\|_F$	$\|F_2\|_F$	$\kappa_2(Y)$
5.1	0.0502	0.0819	20.67	0.0501	0.0818	20.66	0.0488	0.0797	963.04
5.2	26.58	99.68	10.19	21.31	72.23	177.05	19.36	70.89	$4.9 \cdot 10^3$
5.3	0.1364	0.0339	9.25	0.1342	0.0245	9.25	0.1343	0.0224	$1.2 \cdot 10^4$
5.4	1.8593	1.8560	21.35	1.8586	1.8554	21.24	1.4333	1.4114	$1.4 \cdot 10^5$
$\beta = 10$									
	$\alpha = 1$			$\alpha = 0.5$			$\alpha = 0$		
P.	$\|F_1\|_F$	$\|F_2\|_F$	$\kappa_2(Y)$	$\|F_1\|_F$	$\|F_2\|_F$	$\kappa_2(Y)$	$\|F_1\|_F$	$\|F_2\|_F$	$\kappa_2(Y)$
5.1	0.0502	0.0819	20.67	0.0502	0.0819	20.67	0.0488	0.0797	963.04
5.2	26.81	99.55	10.21	31.34	84.08	35.52	19.36	70.89	$4.9 \cdot 10^3$
5.3	0.1359	0.0327	9.25	0.1347	0.0284	9.25	0.1343	0.0224	$1.2 \cdot 10^4$
5.4	1.8710	1.8674	23.82	1.8708	1.8673	23.81	1.4333	1.4114	$1.4 \cdot 10^5$
$\beta = 100$									
	$\alpha = 1$			$\alpha = 0.5$			$\alpha = 0$		
P.	$\|F_1\|_F$	$\|F_2\|_F$	$\kappa_2(Y)$	$\|F_1\|_F$	$\|F_2\|_F$	$\kappa_2(Y)$	$\|F_1\|_F$	$\|F_2\|_F$	$\kappa_2(Y)$
5.1	0.0502	0.0819	20.67	0.0502	0.0819	20.67	0.0488	0.0797	963.04
5.2	26.87	99.52	10.22	30.11	95.67	11.41	19.36	70.89	$4.9 \cdot 10^3$
5.3	0.1358	0.0324	9.25	0.1356	0.0317	9.25	0.1343	0.0224	$1.2 \cdot 10^4$
5.4	1.8721	1.8685	24.01	1.8721	1.8685	24.01	1.4333	1.4114	$1.4 \cdot 10^5$

Table 1: **Numerical results for Algorithm 4.1**

Comparison of the Proposed Algorithms with Brahma-Datta Algorithm

Both these algorithms have the same efficiency in the sense that operation count are of the same order of magnitudes.

Below we compare the Brahma-Datta algorithm with the proposed algorithm for actual performances, taking $\alpha = 1$ and $\beta = 1$. The notations used there are:

- IV = Values of the concerned quantities with the initial parameter Γ.
- FV = Values of the concerned quantities with the final parameter Γ (obtained by the algorithm).
- RE − Relative change (in percent): $= 100 \times \frac{(IV - FV)}{IV}$

Problem		Brahma-Datta's Method			Alg. 4.1 with $\alpha = 1$ and $\beta = 1$		
		$\|F_1\|_F$	$\|F_2\|_F$	$\kappa_2(Y)$	$\|F_1\|_F$	$\|F_2\|_F$	$\kappa_2(Y)$
5.1	IV.	0.0554	0.0905	$4.83 \cdot 10^3$	0.0554	0.0905	$4.83 \cdot 10^3$
	FV.	2.5451	4.1538	263.15	0.0502	0.0819	20.67
	RE.	−4490.50	−4490.50	94.55	9.47	9.47	99.57
5.2	IV.	74.98	130.71	$6.98 \cdot 10^3$	74.98	130.71	$6.98 \cdot 10^3$
	FV.	59.44	161.54	232.99	26.58	99.68	10.19
	RE.	20.72	−23.58	96.66	64.55	23.74	99.85
5.3	IV.	0.2621	0.1075	$2.91 \cdot 10^3$	0.2621	0.1075	$2.91 \cdot 10^3$
	FV.	0.2799	0.2187	$2.22 \cdot 10^9$	0.1364	0.0339	9.25
	RE.	−6.80	−103.39	$-7.62 \cdot 10^7$	47.97	68.47	99.68
5.4	IV.	$2.12 \cdot 10^8$	$6.45 \cdot 10^8$	$2.76 \cdot 10^{10}$	$2.12 \cdot 10^8$	$6.45 \cdot 10^8$	$2.76 \cdot 10^{10}$
	FV.	$5.3 \cdot 10^5$	$1.19 \cdot 10^6$	$1.10 \cdot 10^9$	$4.0 \cdot 10^5$	$9.1 \cdot 10^5$	$2.56 \cdot 10^8$
	RE.	99.75	99.81	96.02	99.81	99.86	99.07

Table 2: **Comparison of Algorithm 4.1 with Brahma-Datta's Method**

The results in Table 2 show that the proposed algorithm performed better in almost all cases.

Comparison of System Responses

Problem 5.5 below is used to demonstrate the robustness of the proposed algorithm under small perturbations of the stiffness matrix K. Comparisons are made of systems responses both for open-loop and closed-loop systems as **(Figure 1)** well as for closed-loop systems under different perturbations (**Figure 2**).

Problem 5.5 *In this example, $M, D, K,$ and B are taken as follows:*

$$M = I_3, \quad D = \begin{bmatrix} 12.5 & 10.0 & 0 \\ 10.0 & 8.5 & 2.0 \\ 0 & 2.0 & 12.5 \end{bmatrix}, \quad K = \begin{bmatrix} 16 & 12 & 0 \\ 12 & 13 & 4 \\ 0 & 4 & 29 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{bmatrix}.$$

Then the quadratic pencil $P(\lambda) = \lambda^2 M + \lambda D + K$ has 6 open-loop eigenvalues $\{-19.4889, -9.4839, -0.0586 \pm 1.4429i, -1.3620, -3.0479\}$. We replace the two open-loop complex eigenvalues $\{-0.0586 \pm 1.4429i\}$ by $\{-0.5 \pm 1.4429i\}$. The other eigenvalues are kept unchanged.

Let

$$\Gamma = 10^2 \cdot \begin{bmatrix} -8i & 8i \\ 3+8i & 3-8i \end{bmatrix}. \tag{65}$$

By Theorem 4.3, we obtain the feedback matrices

$$F_1 = \begin{bmatrix} 0.3665 & -0.3917 \\ -0.4440 & 0.4803 \\ 0.0679 & -0.0746 \end{bmatrix} \quad \text{and} \quad F_2 = \begin{bmatrix} 0.1860 & -0.3485 \\ -0.0057 & 0.3017 \\ -0.0377 & -0.0106 \end{bmatrix}.$$

The eigenvalues of the corresponding close-loop system are $\{-19.4889, -9.4839, -0.5 \pm 1.4429i, -1.3620, -3.0479\}$.

Then we compute the system responses of the open-loop and the close-loop systems with the feedback matrices F_1 and F_2 obtained by Theorem 4.3. Figure 1 depicts the base 10 logarithm of the norm of the system responses over the given time period. The initial condition is $\mathbf{z}(0) = 0.01 \cdot \mathbf{1}_{2n}$, where $\mathbf{1}_{2n} = (1, \dots, 1)^T \in \mathbb{R}^{2n}$. *We observe from Figure 1 that the system response behavior of the close-loop system with the feedback matrices F_1 and F_2 obtained by Theorem 4.3 is better than that of the original open-loop system which was expected.*

Next, we perturb the stiffness matrix K to $K + cK$ with the parameter c varying from -0.1 to 0.1, and keep the matrices M and D, and B fixed.

We then compute the system responses of the open-loop system and the perturbed close-loop systems with the feedback matrices F_1 and F_2 obtained by Algorithm 4.1 (we set $\alpha = \beta = 1$). Figure 2 shows the base 10 logarithm of the norm of the system responses over the defined time period for different values of c. The initial condition is $\mathbf{z}(0) = 0.01 \cdot \mathbf{I}_{2n}$. *We can see from Figure 1 that the system responses of the perturbed close-loop system with the feedback matrices F_1 and F_2 obtained by Algorithm 4.1 are all insensitive to perturbation and successfully tend to the steady state, showing the robustness of the algorithm.*

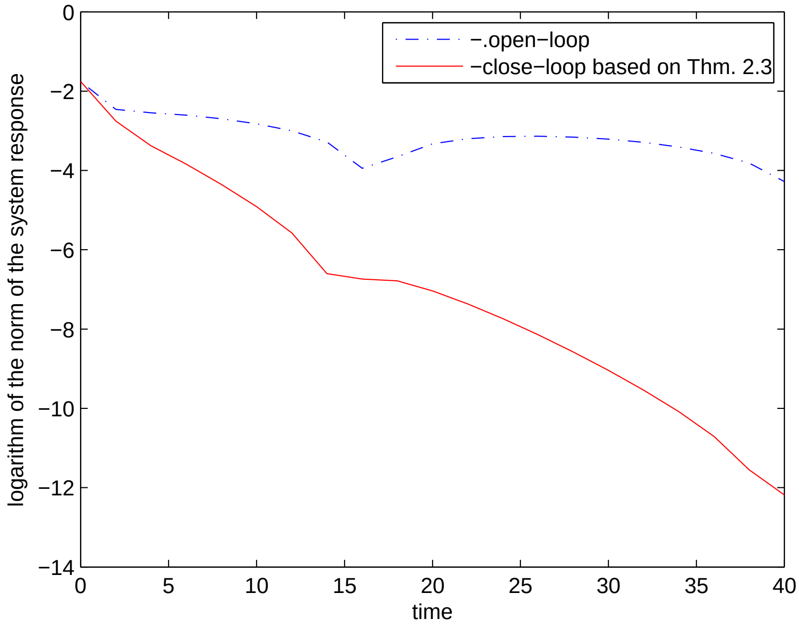

Figure 1: **Comparison of the norms of the system responses for Example 5.5**

6 Summary and Conclusions

A new optimization-based method for robust design of active vibration control in structures modeled by a system of second-order differential equations is proposed. The method can be

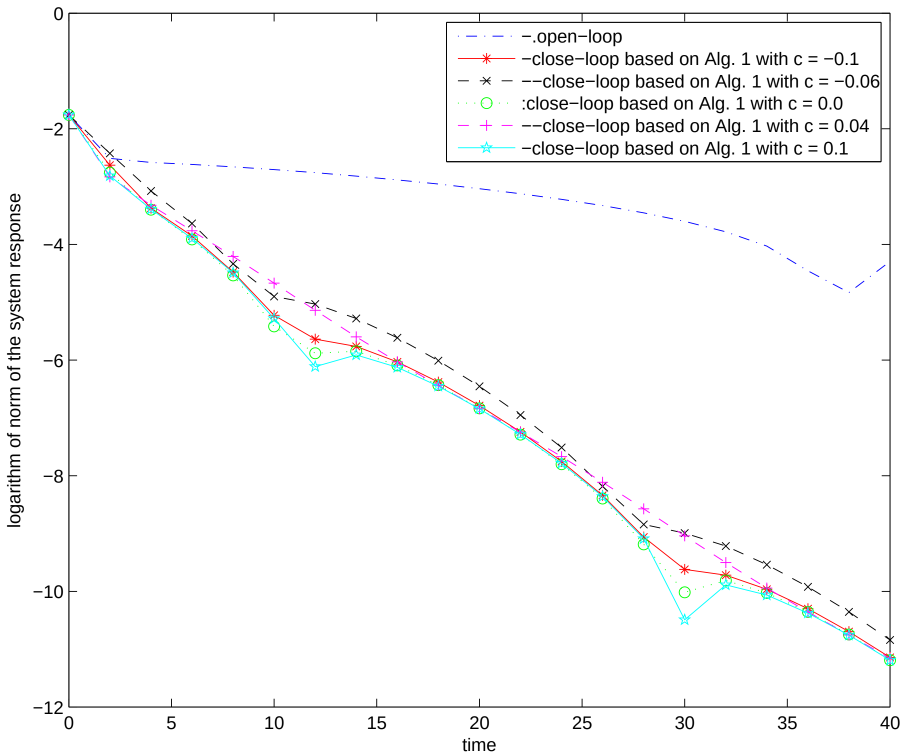

Figure 2: **Comparison of the norm of the system responses for Example 5.5 with perturbed** K

implemented with the knowledge of only a small number of eigenvalues that need to be reassigned to control vibration, and the associated eigenvectors. The no spill-over of the unassigned eigenvalues and eigenvectors is established by means of a mathematical theory. Furthermore, the method is implemented in the original second-order system itself without requiring transformation into the standard first-order state-space control system, and any reduction of the order of the original model. These attractive practical features make the method suitable for even large-scale practical applications. The natural mathematical models of vibrating structures are, however, distributed parameter systems. Though there now exist some methods for partial eigenvalue assignment in distributed parameter systems, methods for robust active vibration control design in distributed parameter systems do not exist. A natural extension of the proposed method to such systems is presently being investigated. Also, the possibility of using the present scheme in conjunction with some passive control device in robust active vibration control of real-life models, such as, beams, plates, etc., that have infinite degree of freedom, is also being currently explored.

References

[1] M. Balas, *Trends in large space structure control theory: Fondest hopes, wildest dreams*, IEEE Trans. Automat. Control, 27 (1982), pp. 522–535.

[2] A. Bhaya and C. Desoer, *On the design of large flexible space structures*, IEEE Trans. Automat. Control, 30 (1985), pp. 1118–1120.

[3] S. Brahma and B. N. Datta, *A norm-minimizing parametric algorithm for quadratic partial eigenvalue assignment via Sylvester equation*, Proc. European Control Conference, Kos, Greece, July, 2007, pp. 1–6

[4] S. Brahma and B. N. Datta, *An optimization approach for minimum norm and robust partial quadratic eigenvalue assignment problems for vibrating structures*, to appear in *J. Sound and Vibration.*

[5] R. Byers and S. G. Nash, *Approaches to robust pole assignment* , Int. J. Control, 49 (1989), pp. 97–117.

[6] J. Carvalho and B. N. Datta, *A new algorithm for generalized Sylvester equation and its applications to state and velocity estimations in vibrating systems*, to appear in *Numerical Linear Algebra with Applications.*

[7] R. K. Cavin III and S. P. Bhattacharya, *Robust and well-conditioned eigenstructure assignment via Sylvester's equation*, Optimal control Application & Methods, 4 (1983), pp. 205–212.

[8] E. K. Chu, *Pole assignment for second-order systems*, Mechanical Systems and Signal Processing, 16 (2002), pp. 39–59.

[9] E. K. Chu and B. N. Datta, *Numerically robust pole assignment for second-order systems*, Int. J. Control, 64 (1996), pp. 1113–1127.

[10] B. N. Datta, *Numerical Linear Algebra and Applications*, SIAM Publications, Philadelphia, 2009 (First edition was published by Brooks/Cole Publishing Company, 1995).

[11] B. N. Datta, *Numerical Methods for Linear Control Systems Design and Analysis*, Elsevier Academic Press, New York, 2003.

[12] B. N. Datta, S. Elhay, and Y. M. Ram, *Orthogonality and partial pole assignment for the symmetric definite quadratic pencil*, Linear Algebra and its Applications, 257 (1997), pp. 29–48.

[13] B. N. Datta, S. Elhay, and Y. M. Ram, *An algorithm for the partial multi-input pole assignment of a second-order control system*, Proceedings of the IEEE Conference on Decision and Control, Kobe, Japan, December 1996, pp. 2025–2029.

[14] B. N. Datta, S. Elhay, Y. M. Ram, and D. R. Sarkissian, *Partial eigenstructure assignment for the quadratic pencil*, J. Sound Vibration, 230 (2000), pp. 101–110.

[15] B. N. Datta, W. W. Lin, and J. N. Wang, *Robust partial pole assignment for vibrating structures with aerodynamic effect*, IEEE Transactions on Automatic Control, 51-12 (2006), pp. 1979–1984.

[16] B. N. Datta, Y. M. Ram and D. R. Sarkissian, *Spectrum Modification for Gyroscopic Systems*, ZAMM : Z. Angew. Math. Mech. 82 (2002), pp. 191–200.

[17] B. N. Datta and Y. Saad, *Arnoldi methods for large Sylvester-like observer matrix equations and an associated algorithm for partial spectrum assignment*, Lin. Alg. Appl., 154-156 (1991), pp. 225–244.

[18] B. N. Datta and D. R. Sarkissian, *Feedback control in distributed parameter gyroscopic systems: A solution of the partial eigenvalue assignment*, Mechanical Systems and Signal Processing, 16-1 (2002), pp. 3–17.

[19] B. N. Datta and D. R. Sarkissian, *Partial eigenvalue assignment in linear systems: existence, uniqueness and numerical solution*, Proceedings of the Mathematical Theory of Networks and Systems (MTNS), Notre Dame, August 2002.

[20] B. N. Datta and D. R. Sarkissian, *Theory and computations of some inverse eigenvalue problems for the quadratic pencil*, Contemporary Mathematics, 280 (2001), pp. 221–240.

[21] M. Friswell and J. Mottershead, *Finite Element Model Updating in Structural Dynamics*, Klawer Publishing Company, 1995.

[22] G. H. Golub and C. F. Van Loan, *Matrix Computations*, Third Edition, The John Hopkins University Press, Baltimore and London, 1996.

[23] D. Inman, *Vibration with Control*, Wiley Publishing Co., 2007.

[24] S. Joshi, *Control of Large Flexible Space Structures*, Lecture Notes in Control and Inform. Sci. 131, Berlin and New York, Springer-Verlag, 1989.

[25] L. H. Keel and S. P. Bhattacharyya, *State space design of low-order stabilizers*, IEEE Trans. Automat. Control, 35 (1990), pp. 182–186.

[26] L. H. Keel, J. A. Fleming and S. P. Bhattacharya, *Minimum norm pole assignment via Sylvester's Equation*, Contemporary Mathematics, 47 (1985).

[27] L. Meirovitch and H. Baruh, *Optimal control of damped flexible gyroscopic systems*, Journal of Guidance, Control, and Dynamics, 4 (1981), pp. 157–163.

[28] L. Meirovitch, H. Baruh, and H. Oz, *A comparison of control techniques for large flexible systems*, Journal of Guidance, Control, and Dynamics, 6 (1983), pp. 302–310.

[29] J. E. Mottershead, M. G. Tehrani, and Y. M. Ram, *Assignment of eigenvalue sensitivities from receptance measurements*, Mechanical Systems and Signal Processing, Special issue on "*Inverse Problems in Mechanical Systems and Signal Processing*", (in press), 2009.

[30] N. K. Nichols and J. Kautsky, *Robust eigenstructure assignment in quadratic matrix polynomials: nonsingular case*, SIAM J. Matrix Anal. Appl., 23 (2001), pp. 77–102.

[31] J. Nocedal and S. J. Wright, *Numerical Optimization*, Springer, New York, 1999.

[32] B. Porter and R. Crossley, *Model Control: Theory and Applications*, Barnes & Noble, New York, 1972.

[33] J. Qian and S. Xu, *Robust partial eigenvalue assignment problem for the second-order system*, Journal of Sound and Vibration, 282 (2005), pp. 937–948.

[34] Y. Ram and S. Elhay, *Pole Assignment in Vibratory Systems by Multi-input Control*, Journal of Sound and Vibration, 230 (2000), pp. 309–321.

[35] J. T. Sawicki and G. Genta, *Modal uncoupling of damped gyroscopic systems*, Journal of Sound and Vibration, 244 (2001), pp. 431–451.

[36] E. De Souza and S. P. Bhattacharyya, *Controllability, observability and the solution of* $AX - XB = C$, Linear Algebra Appl., Vol. 39, pp. 101–110, 1981.

[37] J. G. Sun, *Backward perturbation analysis of certain characteristic subspaces*, Numer. Math., 65 (1993), 357–382.

[38] F. Tisseur and K. Meerbergen, *The quadratic eigenvalue problem*, SIAM Rev., 43 (2001), pp. 235–286.

[39] A. Varga, *Robust pole assignment via Sylvester equation based state feedback parameterization*, Proceedings of the 2000 IEEE International Symposium on Computer-Aided Control System Design, Anchorage, Alaska, USA September 25–27, 2000.

[40] A. Varga, *A numerically reliable approach to robust pole assignment for descriptor systems*, Future Generation Computer Systems, 19 (2003), pp. 1221–1230.