A Multi-Step Hybrid Method for Multi-Input Partial Quadratic Eigenvalue Assignment with Time Delay

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April 14, 2012

Abstract

A hybrid method was given by Ram, Mottershead, and Tehrani [*Linear Algebra Appl., 434 (2011), pp. 1689–1696*] for solving the partial quadratic eigenvalue assignment problem of single-input vibratory systems. In this paper, we consider the partial quadratic eigenvalue assignment problem of multi-input vibratory systems. We solve the multi-input partial quadratic eigenvalue assignment problem by a multi-step hybrid method using both the system matrices and the receptance measurements. Our method can assign the partial expected eigenvalues and keep the no spillover property. We also extend our method to the case when there exists time delay between measurements of state and actuation of control. Numerical tests show the effectiveness of our method.

Keywords. Vibrating control, partial quadratic eigenvalue assignment, time delay, receptance measurements

AMS subject classification. *65F18, 93B55, 93C15*

1 Introduction

The vibrations of various vibratory structures (e.g., buildings, bridges, and highways) are often governed by the following second-order differential equation:

$$M\ddot{\mathbf{x}}(t) + D\dot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{f}(t), \tag{1}$$

where M, D, K denote, respectively, the mass, damping, and stiffness matrices, which are all real symmetric matrices of order n, $\mathbf{x}(t)$ is n-vector dependent on the time t, and $\mathbf{f}(t)$ represents an external force. In many applications, M is positive definite, D and K positive semidefinite.

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It is well-known that the stability and dynamic analysis of the system (1) is vitally related to the solution of the quadratic eigenvalue problem [20]

$$P(\lambda)\mathbf{x} := (\lambda^2 M + \lambda D + K)\mathbf{x} = \mathbf{0}, \tag{2}$$

which has $2n$ eigenpairs $\{(\lambda_i, \mathbf{x}_i)\}_{i=1}^{2n}$.

However, if some of the eigenvalues $\{\lambda_i\}_{i=1}^{2n}$ have positive real parts or negative real parts but very close to zeros, then the the homogeneous equation of (1) is unstable. On the other hand, when the external force $\mathbf{f}(t)$ has a frequency near one of the eigenvalues $\{\lambda_i\}_{i=1}^{2n}$, a resonance phenomenon will appear. This is unwanted in many vibration structures, e.g., buildings, bridges, and highways, etc.

To avoid the unstability and unwanted resonance for the vibrating structures, an active vibration control aims to find a feedback control force $\mathbf{f}(t)$ such that the few unwanted eigenvalues are replaced by the desired ones. In general, the feedback control force $\mathbf{f}(t)$ has a form of

$$\mathbf{f}(t) = B\mathbf{u}(t) \quad \text{with} \quad \mathbf{u}(t) = F^T\dot{\mathbf{x}}(t) + G^T\mathbf{x}(t), \tag{3}$$

where $B \in \mathbb{R}^{n\times m}$ is a control matrix, $\mathbf{u}(t) \in \mathbb{R}^m$ is the control m-vector, and $F, G \in \mathbb{R}^{n\times m}$ are feedback matrices. In this case, we obtain the following closed-loop equation

$$M\ddot{\mathbf{x}}(t) + D\dot{\mathbf{x}}(t) + K\mathbf{x}(t) = B(F^T\dot{\mathbf{x}} + G^T\mathbf{x}(t)), \tag{4}$$

which leads to the closed-loop quadratic eigenvalue problem

$$P_c(\lambda)\mathbf{y} := \big(\lambda^2 M + \lambda(D - BF^T) + (K - BG^T)\big)\mathbf{y} = \mathbf{0}. \tag{5}$$

The partial quadratic eigenvalue assignment problem (PQEAP) is to find the feedback matrices $F, G \in \mathbb{R}^{n\times m}$ such that the partial unwanted eigenvalues of the open-loop pencil $P(\lambda)$ are assigned to the prescribed ones and the remaining eigenvalues and associated eigenvectors are kept unchanged, i.e., the no spill-over property is preserved. There is a large literature on numerical approaches for the PQEAP (see for instance [2, 3, 4, 5, 6, 7, 13]). All these approaches require only the availability of the system matrices, a few unwanted eigenvalues of the open-loop pencil $P(\lambda)$ and their associated eigenvectors.

In this paper, we propose a multi-step hybrid method for solving the PQEAP by combining the system matrices with the measured receptances. This is motivated by the multi-step method for the PQEAP proposed in [13] and recent developments on the method of receptances in active vibration control [11, 12, 14, 15, 18, 19]. In particular, in [13], Ram and Elhay gave a multi-step method for solving the PQEAP with multi-input control (i.e., $m > 1$). In [15], based on the measured receptances and the system matrices, Ram, Mottershead, and Tehrani presented a computational method for the PQEAP with single-input state feedback (i.e., $m = 1$). Since the receptance is measurable in applications [8], the combination of the system matrices and the measured receptances, together with the multi-step method in [13], gives rise to a multi-step hybrid method for solving the multi-input PQEAP. By using the measured receptances, the proposed multi-step hybrid method can reduce the total computational cost over the multi-step method in [13]. We also apply our multi-step hybrid method to the PQEAP with time delay, which is not discussed in [1], [13], [14] or [15].

This paper is organized as follows. In Section 2 we recall the preliminary results on the parameterized solution to the PQEAP. In Sections 3 we propose a multi-step hybrid method for solving the PQEAP by the combination of the system matrices and the measured receptances. In Section 4 we extend our multi-step hybrid method to the PQEAP with time delay. The illustrative numerical examples are also reported.

2 Parameterized Solution

Let A^T denote the transpose of a matrix A. Denote by $\|\cdot\|$ the Euclidean vector norm or its induced matrix norm. Suppose that we assign the partial eigenvalues $\{\lambda_k\}_{k=1}^{p}$ ($p \ll 2n$) of the open-loop pencil $P(\lambda)$ to the prescribed p complex numbers $\{\mu_k\}_{k=1}^{p}$. In general, the partial eigendata $\{(\lambda_k, \mathbf{x}_k)\}_{k=1}^{p}$ may be measured or estimated by experiments [9]. The PQEAP aims to find the feedback matrices $F, G \in \mathbb{R}^{n\times m}$ such that the multi-input closed-loop system (4) has p new eigenvalues $\{\mu_k\}_{k=1}^{p}$ and the $2n-p$ eigenpairs $\{(\lambda_k, \mathbf{x}_k)\}_{k=p+1}^{2n}$. In what follows, we suppose that $\{\mu_k\}_{k=1}^{p} \cap \{\lambda_k\}_{k=1}^{2n} = \emptyset$, $\{\lambda_k\}_{k=1}^{p} \cap \{\lambda_k\}_{k=p+1}^{2n} = \emptyset$, the matrix B is full column rank, and $(P(\lambda), B)$ is partially controllable with respect to $\{\lambda_k\}_{k=1}^{p}$, i.e., $\mathrm{rank}(P(\lambda_k), B) = n$, for $k = 1, \ldots, p$. Define

$$\begin{aligned} \Lambda_1 &= \mathrm{diag}(\lambda_1, \ldots, \lambda_p), \quad X_1 = [\mathbf{x}_1, \ldots, \mathbf{x}_p], \\ \Lambda_2 &= \mathrm{diag}(\lambda_{p+1}, \ldots, \lambda_{2n}), \quad X_2 = [\mathbf{x}_{p+1}, \ldots, \mathbf{x}_{2n}]. \end{aligned} \tag{6}$$

On the parameterized solution to the PQEAP, we have the following result.

Lemma 2.1 *[1, Theorem 4.3] Given a self-conjugate set of complex numbers* $\{\mu_k\}_{k=1}^{p}$*.*

(a) Let the feedback matrices F *and* G *be given by*

$$F = MX_1\Phi^T \quad \text{and} \quad G = (MX_1\Lambda_1 + DX_1)\Phi^T, \tag{7}$$

where $\Phi \in \mathbb{C}^{m\times p}$ *is arbitrary. Then we have*

$$MX_2\Lambda_2^2 + (D - BF^T)X_2\Lambda_2 + (K - BG^T)X_2 = 0.$$

i.e., the no spill-over property is preserved.

(b) Choose $\Phi \in \mathbb{C}^{m\times p}$ *such that* $\Phi Z = \Gamma$*, where* $\Gamma = [\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_p] \in \mathbb{C}^{m\times p}$ *is arbitrary nonzero matrix such that if* $\mu_j = \bar{\mu}_k$*, then* $\boldsymbol{\gamma}_j = \bar{\boldsymbol{\gamma}}_k$*, and* Z *is the solution to the Sylvester equation*

$$\Lambda_1 Z - Z\Sigma = -X_1^T B\Gamma, \tag{8}$$

where $\Sigma = \mathrm{diag}(\mu_1, \ldots, \mu_p)$*. Then the feedback matrices* F *and* G *defined in* (7) *are real and the* p *complex numbers* $\mu_1, \ldots, \mu_p$ *are in the spectrum of the closed-loop system* (4)*.*

3 A Multi-Step Hybrid Method for the Partial Quadratic Eigenvalue Assignment

In this section, we propose a multi-step hybrid method for the PQEAP. We note that the closed-loop control system defined in (4) can be written as

$$M\ddot{\mathbf{x}}(t) + D\dot{\mathbf{x}}(t) + K\mathbf{x}(t) = \sum_{k=1}^{m} \mathbf{b}_k\big(\mathbf{f}_k^T\dot{\mathbf{x}}(t) + \mathbf{g}_k^T\mathbf{x}(t)\big), \tag{9}$$

where $\mathbf{b}_k$, $\mathbf{f}_k$, and $\mathbf{g}_k$ are the kth columns of B, F, and G, respectively. Define

$$\eta_{jk} := \lambda_j + \frac{k}{m}(\mu_j - \lambda_j), \tag{10}$$

for $j = 1, \ldots, p$ and $k = 0, 1, \ldots, m$. We note that $\eta_{j0} = \lambda_j$ and $\eta_{jm} = \mu_j$ for $j = 1, \ldots, p$. Let

$$D_k := D - \sum_{i=1}^{k-1} \mathbf{b}_i\mathbf{f}_i^T \quad \text{and} \quad K_k := K - \sum_{i=1}^{k-1} \mathbf{b}_i\mathbf{g}_i^T, \tag{11}$$

for $k = 2, \ldots, m$ and $D_1 := D$ and $K_1 := K$. Then the PQEAP is solved if we can find a solution to the following multi-step problem.

Problem 1. *Given M, D, K, B, Λ_1, X_1, and a self-conjugate set of $\{\mu_j\}_{j=1}^p$. Let $\{D_k, K_k\}_{k=1}^m$ and $\{\eta_{jk}\}$ be defined in (11) and (10), respectively. For $k = 1, \ldots, m$, find the two feedback vectors $\mathbf{f}_k$ and $\mathbf{g}_k$ successively such that the single-input closed-loop feedback control system*

$$M\ddot{\mathbf{x}}(t) + D_k\dot{\mathbf{x}}(t) + K_k\mathbf{x}(t) = \mathbf{b}_k\big(\mathbf{f}_k^T\dot{\mathbf{x}}(t) + \mathbf{g}_k^T\mathbf{x}(t)\big) \tag{12}$$

has the desired eigenvalues $\{\eta_{jk}\}_{j=1}^p$ and the eigenpairs $\{(\lambda_j, \mathbf{x}_j)\}_{j=p+1}^{2n}$.

For $k = 1, \ldots, m$, define the feedback vectors $\mathbf{f}_k$ and $\mathbf{g}_k$ by

$$\mathbf{f}_k := MX_1\mathbf{w}_k \quad \text{and} \quad \mathbf{g}_k := (MX_1\Lambda_1 + DX_1)\mathbf{w}_k, \tag{13}$$

where $\mathbf{w}_k \in \mathbb{C}^{p\times 1}$ is arbitrary. Then, by Lemma 2.1, $\{(\lambda_j, \mathbf{x}_j)\}_{j=p+1}^{2n}$ are eigenpairs of the single-input closed-loop system (12).

In the following, we present a hybrid method for solving Problem 1. For any $s \in \mathbb{C}$, the receptance matrix

$$H(s) := (s^2M + sD + K)^{-1}$$

can be accessible by physical measurements [8]. Let $\mathbf{b}_0 = \mathbf{0} \in \mathbb{R}^n$, $\mathbf{f}_0 = \mathbf{0} \in \mathbb{R}^n$, and $\mathbf{g}_0 = \mathbf{0} \in \mathbb{R}^n$. Starting from the first step, based on the known $\{\mathbf{f}_i\}_{i=0}^{k-1}$ and $\{\mathbf{g}_i\}_{i=0}^{k-1}$, we may find the feedback vectors $\mathbf{f}_k$ and $\mathbf{g}_k$ successively such that the single-input closed-loop feedback control system (12) has the desired eigenvalues $\{\eta_{j,k}\}_{j=1}^p$ and the eigenpairs $\{(\lambda_j, \mathbf{x}_j)\}_{j=p+1}^{2n}$ for $k = 1, \ldots, m$.

We now develop a hybrid method in the kth step. Let $H_k(s)$ be the receptance matrix related to the single-input closed-loop feedback control system (12). By the Sherman-Morrison formula [10, 17], we get

$$H_k(s) = H_{k-1}(s) + \frac{H_{k-1}(s)\mathbf{b}_k(\mathbf{g}_k^T + s\mathbf{f}_k^T)H_{k-1}(s)}{1 - (\mathbf{g}_k^T + s\mathbf{f}_k^T)H_{k-1}(s)\mathbf{b}_k}, \tag{14}$$

where $H_0(s) = H(s)$. Notice that $H_k(s)$ gets unbounded as $s \to \eta_{jk}$ for $j = 1, \ldots, p$. It follows from (14) that

$$(\mathbf{g}_k^T + \eta_{jk}\mathbf{f}_k^T)H_{k-1}(\eta_{jk})\mathbf{b}_k = 1, \quad j = 1, \ldots, p$$

or

$$G_k \begin{bmatrix} \mathbf{f}_k \\ \mathbf{g}_k \end{bmatrix} = \mathbf{e}_p, \tag{15}$$

where

$$G_k := \begin{bmatrix} \eta_{1k}\mathbf{b}_k^T H_{k-1}^T(\eta_{1k}) & \mathbf{b}_k^T H_{k-1}^T(\eta_{1k}) \\ \eta_{2k}\mathbf{b}_k^T H_{k-1}^T(\eta_{2k}) & \mathbf{b}_k^T H_{k-1}^T(\eta_{2k}) \\ \vdots & \\ \eta_{pk}\mathbf{b}_k^T H_{k-1}^T(\eta_{pk}) & \mathbf{b}_k^T H_{k-1}^T(\eta_{pk}) \end{bmatrix}, \quad \mathbf{e}_p := \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^p. \tag{16}$$

Therefore, we have the following result for the kth step of Problem 1. Our proof is sparked by Ref. [13].

Theorem 3.1 *For $k = 1, \ldots, m$, let the feedback vectors $\mathbf{f}_k$ and $\mathbf{g}_k$ defined by (13) with $\mathbf{w}_k$ determined by*

$$A_k\mathbf{w}_k := G_k \begin{bmatrix} MX_1 \\ MX_1\Lambda_1 + DX_1 \end{bmatrix} \mathbf{w}_k = \mathbf{e}_p, \tag{17}$$

where G_k and $\mathbf{e}_p$ are defined as in (16). Then $\mathbf{f}_k$ and $\mathbf{g}_k$ are real and the single-input closed-loop feedback control system (12) has the desired eigenvalues $\{\eta_{jk}\}_{j=1}^p$ and the eigenpairs $\{(\lambda_j, \mathbf{x}_j)\}_{j=p+1}^{2n}$.

Proof: Without loss of generality, we assume that $\{\mathbf{f}_i\}_{i=0}^{k-1}$ and $\{\mathbf{g}_i\}_{i=0}^{k-1}$ are already available. Then, the quadratic eigenvalue problem associated with the closed-loop system (12) is given by

$$\big(\tau^2 M + \tau(D_k - \mathbf{b}_k\mathbf{f}_k^T) + (K_k - \mathbf{b}_k\mathbf{g}_k^T)\big)\mathbf{y} = \mathbf{0}. \tag{18}$$

Since the feedback vectors $\mathbf{f}_k$ and $\mathbf{g}_k$ are defined by (13), by Lemma 2.1, it is obvious that $\{(\lambda_j, \mathbf{x}_j)\}_{j=p+1}^{2n}$ are eigenpairs of the closed-loop system (12).

We now determine $\mathbf{w}_k$ such that $\{\eta_{jk}\}_{j=1}^p$ are eigenvalues of the closed-loop system (12). By (18), for $j = 1, \ldots, p$, an eigenpair $(\eta_{jk}, \mathbf{y}_{jk})$ satisfies

$$\big(\eta_{jk}^2 M + \eta_{jk}(D_k - \mathbf{b}_k\mathbf{f}_k^T) + (K_k - \mathbf{b}_k\mathbf{g}_k^T)\big)\mathbf{y}_{jk} = \mathbf{0}, \tag{19}$$

i.e.,

$$(\eta_{jk}^2 M + \eta_{jk}D_k + K_k)\mathbf{y}_{jk} = \mathbf{b}_k(\eta_{jk}\mathbf{f}_k^T + \mathbf{g}_k^T)\mathbf{y}_{jk}.$$

Since $\delta_{jk} := (\eta_{jk}\mathbf{f}_k^T + \mathbf{g}_k^T)\mathbf{y}_{jk}$ is a scalar quantity, we can find an eigenvector $\hat{\mathbf{y}}_{jk}$ of the closed-loop system (12) corresponding to η_{jk} by solving

$$(\eta_{jk}^2 M + \eta_{jk}D_k + K_k)\hat{\mathbf{y}}_{jk} = \mathbf{b}_k,$$

where $\hat{\mathbf{y}}_{jk} = \delta_{jk}^{-1}\mathbf{y}_{jk}$, which gives rise to

$$\hat{\mathbf{y}}_{jk} = H_{k-1}(\eta_{jk})\mathbf{b}_k. \tag{20}$$

By (13), we have

$$\delta_{jk} = (\eta_{jk}\mathbf{f}_k^T + \mathbf{g}_k^T)\mathbf{y}_{jk} = \mathbf{w}_k^T(\eta_{jk}X_1^T M + \Lambda_1 X_1^T M + X_1^T D)\mathbf{y}_{jk}$$

or

$$\hat{\mathbf{y}}_{jk}^T(\eta_{jk}MX_1 + MX_1\Lambda_1 + DX_1)\mathbf{w}_k = 1, \quad j = 1, \ldots, p. \tag{21}$$

This, together with (20), yields (17).

Also, the fact that $\mathbf{f}_k$ and $\mathbf{g}_k$ are real can be proved by following the similar proof of [14, Theorem 2]. The proof is complete. □

Remark 3.2 *We point out that for equation (17) to be solvable the eigenvalues* $\{\lambda_j\}_{j=1}^p$ *have to be distinct and* $\{\mu_k\}_{k=1}^p \cap \{\lambda_k\}_{k=1}^{2n} = \emptyset$. *We also see from (20) that* $\{(\mu_j, \hat{\mathbf{y}}_{jm})\}_{j=1}^p$ *are* p *eigenpairs of the closed-loop pencil* $P_c(\lambda)$ *in* (5).

Remark 3.3 *We observe from Theorem 3.1 that, in the kth step, we need* $H_{k-1}(s)$ *to solve* (17)*. Since* $H_0(s) = H(s)$ *is available from the physical tests, it follows from* (14) *that* $H_{k-1}(s)$ *can be computed based on* $H(s)$, $\{\mathbf{f}_i\}_{i=0}^{k-1}$ *and* $\{\mathbf{g}_i\}_{i=0}^{k-1}$.

For demonstration purpose, we state the multi-step hybrid algorithm as follows.

Algorithm I: Multi-Step Hybrid Method for the PQEAP
Inputs:

1. The matrices $M, D, K \in \mathbb{R}^{n\times n}$, where M, D, K are symmetric with M being positive definite.
2. The control matrix $B \in \mathbb{R}^{n\times m}$ $(m \le n)$.
3. A self-conjugate set $\{\lambda_i\}_{i=1}^p$ of the spectrum of $P(\lambda)$ with associated eigenvectors $\{\mathbf{x}_j\}_{j=1}^p$.
4. A prescribed self-conjugate set $\{\mu_i\}_{i=1}^p$ and the measured data $\{H_0(\eta_{jk}) = H(\eta_{jk}) : j = 1, \ldots, p, k = 1, \ldots, m\}$, where $\{\eta_{jk}\}$ is defined in (10).

Outputs:

The real feedback matrices $F = [\mathbf{f}_1, \ldots, \mathbf{f}_m]$ and $G = [\mathbf{g}_1, \ldots, \mathbf{g}_m]$ such that the spectrum of the close-loop system (4) is $\{\mu_1, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n}\}$. Also, the closed-loop feedback control system

$$M\ddot{\mathbf{x}}(t) + D\dot{\mathbf{x}}(t) + K\mathbf{x}(t) = B_k\big(F_k^T\dot{\mathbf{x}}(t) + G_k^T\mathbf{x}(t)\big)$$

has the desired eigenpairs $\{(\eta_{jk}, \hat{\mathbf{y}}_{jk})\}_{j=1}^p$ and $\{(\lambda_j, \mathbf{x}_j)\}_{j=p+1}^{2n}$ for $k = 1, \ldots, m$, where $B_k = [\mathbf{b}_1, \ldots, \mathbf{b}_k]$, $F_k = [\mathbf{f}_1, \ldots, \mathbf{f}_k]$, and $G_k = [\mathbf{g}_1, \ldots, \mathbf{g}_k]$.

Step 1. Form the matrices Λ_1 and X_1 by (6).

Step 2. Set $e_p = [1, \ldots, 1]^T \in \mathbb{R}^p$.

Step 3. Compute $\hat{\mathbf{y}}_{j1}$ by (20). This step needs $O(n^2p)$ operations.

Step 4. Compute $\mathbf{w}_1$ by solving (17). This step requires $O(n^2p + np^2 + p^3)$ flops.

Step 5. Form $\mathbf{f}_1$ and $\mathbf{g}_1$ by (13). This step needs $O(n^2 + np)$ flops.

Step 6. For $k = 2, \ldots, m$

Step 6.1 For $j = 1, \ldots, p$

Step 6.1.1 For $i = 1, \ldots, k-1$, compute $H_i(\eta_{jk})$ successively by (14) using $H_{i-1}(\eta_{jk})$, $\mathbf{f}_i$, and $\mathbf{g}_i$. This step requires $O(n^2k)$ operations.

Step 6.1.2 Compute $\hat{\mathbf{y}}_{jk}$ by (20). This step requires $O(n^2p)$ operations.

Step 6.2 Compute $\mathbf{w}_k$ by solving (17), which requires $O(n^2p + np^2 + p^3)$ flops.

Step 6.3 Form $\mathbf{f}_k$ and $\mathbf{g}_k$ by (13). This step needs $O(n^2 + np)$ flops.

We note that the total computational cost for Algorithm I is $O(n^2m^2p + n^2p + np^2 + p^3)$. In general, $m, p \ll n$. Thus our algorithm requires much lower complexity than the multi-step method in [13], where one has to solve the following mp linear equations

$$(\eta_{jk}^2 M + \eta_{jk} D_k + K_k)\mathbf{z}_{jk} = \mathbf{b}_k, \quad j = 1, \ldots, p, \quad k = 1, \ldots, m,$$

which need $O(n^3mp)$ operations.

Remark 3.4 *In Algorithm I, by using the measured receptance data $\{H(\eta_{jk})\}$ and the system matrices, a sequence of solving (17) is proposed as an alternative to solving the Sylvester equation (8). Algorithm I provides several computational advantages over the method in Lemma 2.1. Our method avoids solving the Sylvester equation (8) whose solution Z is not guaranteed to be nonsingular so that Φ is uniquely determined by $\Phi Z = \Gamma$. Algorithm I does not involve computation of the parametric matrix Γ, whose choice is crucial to the method in Lemma 2.1. One must choose the parameter matrix $\Gamma = [\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_p] \in \mathbb{C}^{m\times p}$ such that if $\mu_j = \bar{\mu}_k$, then $\boldsymbol{\gamma}_j = \bar{\boldsymbol{\gamma}}_k$. Moreover, we have to choose the matrix Γ by trial and error until Z is nonsingular. For Algorithm I, the coefficient matrix A_k in equation (17) are generally nonsingular in practice (see numerical tests below) and equation (17) is well-conditioned since the receptance data $\{H(\eta_{jk})\}_{j=1}^p$ can be measured precisely [8] and $\{H_{k-1}(\eta_{jk})\}_{j=1}^p$ can be computed exactly via (14). Therefore, Algorithm I is stable. Furthermore, our hybrid method can be extended to the case of time delay (see Section 4).*

Remark 3.5 *We observe from Algorithm I that different orders of $\{\mu_j\}_{j=1}^p$ lead to varying feedback matrices F and G. This allows diverse choices of controllers in practice.*

Finally, we give two examples to show that Algorithm I is effective. The numerical experiments were implemented in `MATLAB 7.10` and run on a PC Intel Pentium IV of 3.00 GHZ CPU.

Example 3.6 *Consider the second-order control system* (4) *with $n = 3$ and $m = 2$, where*

$$M = I_3, \quad D = \begin{bmatrix} 2.5 & -0.5 & 0 \\ -0.5 & 2.5 & -2 \\ 0 & -2 & 2 \end{bmatrix}, \quad K = \begin{bmatrix} 10 & -5 & 0 \\ -5 & 25 & -20 \\ 0 & -20 & 20 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

The corresponding open-loop pencil has 6 *eigenvalues:* $\lambda_{1,2} = -0.1512 \pm 1.0372i$, $\lambda_{3,4} = -1.1859 \pm 3.0278i$, *and* $\lambda_{5,6} = -2.1629 \pm 6.1939i$. *The first two eigenvalues* $\lambda_{1,2} = -0.1512 \pm 1.0372i$ *were reassigned to* $\mu_{1,2} = -0.5 \pm 1.0372i$ (i.e., $p = 2$) *and the other eigenvalues and associated eigenvectors were kept unchanged.*

Using Algorithm I to Example 3.6, we get

$$[\mathbf{w}_1, \ldots, \mathbf{w}_m] = \begin{bmatrix} -0.4508 + 0.1886i & -0.2005 + 0.1719i \\ -0.4508 - 0.1886i & -0.2005 - 0.1719i \end{bmatrix},$$

which leads to the following feedback matrices

$$F = \begin{bmatrix} -0.3488 & -0.1745 \\ -0.6253 & -0.3290 \\ -0.6608 & -0.3488 \end{bmatrix}, \quad G = \begin{bmatrix} -0.5372 & -0.3212 \\ 0.0734 & -0.0718 \\ 0.0852 & -0.0719 \end{bmatrix}$$

with $\|F\| = 1.0998$ and $\|G\| = 0.6276$. The errors of the closed-loop eigenvalues and eigenvectors are given by:

$$\begin{cases} \left\| \left(\mu_j^2 M + \mu_j (D - BF^T) + (K - BG^T) \right) \hat{\mathbf{y}}_{jm} \right\| < 5.44 \times 10^{-15}, & 1 \le j \le p, \\ \left\| \left(\lambda_j^2 M + \lambda_j (D - BF^T) + (K - BG^T) \right) \mathbf{x}_j \right\| < 1.72 \times 10^{-13}, & p+1 \le j \le 2n. \end{cases}$$

Example 3.7 **[13]** *Consider the second-order control system* (4) *with* $m = 3$, $p = 4$ *and different* n, *where*

$$M = I_n, \quad D = 0, \quad K = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} I_m \\ 0 \end{bmatrix}.$$

The first $p = 4$ *eigenvalues with smallest absolute values are replaced by* $\mu_{2k-1,2k} = -k \pm \sqrt{-10k}$ *for* $k = 1, 2$ *and the other eigenvalues and associated eigenvectors were preserved.*

Table 1 displays the numerical results for Example 3.7, where `tol1.` and `tol2.` stand for the upper bounds for the errors of the closed-loop eigenvalues and eigenvectors, i.e.,

$$\begin{cases} \left\| \left(\mu_j^2 M + \mu_j (D - BF^T) + (K - BG^T) \right) \hat{\mathbf{y}}_{jm} \right\| < \texttt{tol1.}, & 1 \le j \le p, \\ \left\| \left(\lambda_j^2 M + \lambda_j (D - BF^T) + (K - BG^T) \right) \mathbf{x}_j \right\| < \texttt{tol2.}, & p+1 \le j \le 2n. \end{cases}$$

We can see from Table 1 that, as expected, the unwanted eigenvalues are replaced by new ones with no spillover.

Table 1: Numerical results for Example 3.7

n	$\|F\|$	$\|G\|$	tol1.	tol2.
10	436	1023	4.96×10^{-13}	3.03×10^{-13}
20	4090	10167	2.84×10^{-12}	1.51×10^{-11}
40	42058	109139	2.34×10^{-11}	6.11×10^{-10}
80	453474	1207354	2.55×10^{-10}	2.74×10^{-8}
100	980719	2625513	1.55×10^{-10}	3.48×10^{-8}

4 A Multi-Step Hybrid Method for the Partial Quadratic Eigenvalue Assignment with Time Delay

We consider the following feedback control system with time delay

$$M\ddot{\mathbf{x}}(t) + D\dot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{f}(t - \zeta),$$

where ζ is the input time delay and $\mathbf{f}(t)$ is a state feedback controller defined by (3). The associated closed-loop delayed pencil is given by

$$\widetilde{P}_c(\lambda) := \lambda^2 M + \lambda(D - e^{-\lambda\zeta} BF^T) + (K - e^{-\lambda\zeta} BG^T). \tag{22}$$

The PQEAP with time delay is to find the feedback matrices F and G such that the closed-loop delayed pencil $\widetilde{P}_c(\lambda)$ in (22) has the new prescribed eigenvalues $\{\mu_k\}_{k=1}^p$ and the $2n-p$ eigenpairs $\{(\lambda_k, \mathbf{x}_k)\}_{k=p+1}^{2n}$. We observe that the closed-loop delayed pencil (22) takes the form of

$$\widetilde{P}_c(\lambda) := \lambda^2 M + \lambda\big(D - e^{-\lambda\zeta} \sum_{k=1}^{m} \mathbf{b}_k \mathbf{f}_k^T\big) + \big(K - e^{-\lambda\zeta} \sum_{k=1}^{m} \mathbf{b}_k \mathbf{g}_k^T\big), \tag{23}$$

where $\mathbf{b}_k$, $\mathbf{f}_k$, and $\mathbf{g}_k$ are the kth columns of B, F, and G, respectively. Let $\{\eta_{jk}\}$ be given by (10). Define

$$\widetilde{D}_k := D - e^{-\lambda\zeta} \sum_{i=1}^{k-1} \mathbf{b}_i \mathbf{f}_i^T \quad \text{and} \quad \widetilde{K}_k := K - e^{-\lambda\zeta} \sum_{i=1}^{k-1} \mathbf{b}_i \mathbf{g}_i^T, \tag{24}$$

for $k = 2, \ldots, m$ and $\widetilde{D}_1 := D$ and $\widetilde{K}_1 := K$. Therefore, the PQEAP with time delay is solved if we can find a solution to the following multi-step problem.

Problem 2. *Given M, D, K, B, Λ_1, X_1, ζ, and a self-conjugate set of $\{\mu_j\}_{j=1}^p$. Let $\{\widetilde{D}_k, \widetilde{K}_k\}_{k=1}^m$ and $\{\eta_{jk}\}$ be defined as in (24) and (10), respectively. For $k = 1, \ldots, m$, find the two feedback vectors $\mathbf{f}_k$ and $\mathbf{g}_k$ in turn such that the single-input closed-loop delayed pencil*

$$\widetilde{P}_k(\lambda) := \lambda^2 M + \lambda(\widetilde{D}_k - e^{-\lambda\zeta} \mathbf{b}_k \mathbf{f}_k^T) + (\widetilde{K}_k - e^{-\lambda\zeta} \mathbf{b}_k \mathbf{g}_k^T), \tag{25}$$

has the desired eigenvalues $\{\eta_{jk}\}_{j=1}^p$ and the eigenpairs $\{(\lambda_j, \mathbf{x}_j)\}_{j=p+1}^{2n}$.

Ram, Singh, and Mottershead [16] showed that $2n$ eigenvalues of the single-input closed-loop delayed pencil $\widetilde{P}_k(\lambda)$ can be assigned by the receptance method. In the following, we present a multi-step hybrid method for solving Problem 2 by combining the receptance measurements and the system matrices. For any $s \in \mathbb{C}$, let $\widetilde{H}_k(s)$ be the receptance matrix corresponding to the closed-loop delayed pencil $\widetilde{P}_k(\lambda)$ given in (25). As in Section 3, let $\mathbf{b}_0 = \mathbf{0} \in \mathbb{R}^n$, $\mathbf{f}_0 = \mathbf{0} \in \mathbb{R}^n$, and $\mathbf{g}_0 = \mathbf{0} \in \mathbb{R}^n$. Define $\widetilde{P}_0(\lambda) := P(\lambda)$. Hence, starting from $k = 1$, based on the known $\{\mathbf{f}_i\}_{i=0}^{k-1}$ and $\{\mathbf{g}_i\}_{i=0}^{k-1}$, we may determine the feedback vectors $\mathbf{f}_k$ and $\mathbf{g}_k$ in turn such that the eigenvalues $\{\eta_{j,k-1}\}_{j=1}^{p}$ of the delayed pencil $\widetilde{P}_{k-1}(\lambda)$ are replaced by the new eigenvalues $\{\eta_{j,k}\}_{j=1}^{p}$ of the closed-loop delayed pencil $\widetilde{P}_k(\lambda)$.

We now determine $\mathbf{f}_k$ and $\mathbf{g}_k$ at the kth step. By the Sherman-Morrison formula,

$$\widetilde{H}_k(s) = \widetilde{H}_{k-1}(s) + \frac{e^{-s\zeta}\widetilde{H}_{k-1}(s)\mathbf{b}_k(\mathbf{g}_k^T + s\mathbf{f}_k^T)\widetilde{H}_{k-1}(s)}{1 - e^{-s\zeta}(\mathbf{g}_k^T + s\mathbf{f}_k^T)\widetilde{H}_{k-1}(s)\mathbf{b}_k}, \tag{26}$$

where $\widetilde{H}_0(s) = H(s)$. Notice that $\widetilde{H}_k(s)$ gets unbounded as $s \to \eta_{jk}$ for $j = 1, \ldots, p$. By (26), we have

$$(\mathbf{g}_k^T + \eta_{jk}\mathbf{f}_k^T)\widetilde{H}_{k-1}(\eta_{jk})\mathbf{b}_k = e^{\eta_{jk}\zeta}, \quad j = 1, \ldots, p$$

or

$$\widetilde{G}_k \begin{bmatrix} \mathbf{f}_k \\ \mathbf{g}_k \end{bmatrix} = \widetilde{\mathbf{e}}_p^{(k)},$$

where

$$\widetilde{G}_k := \begin{bmatrix} \eta_{1k}\mathbf{b}_k^T\widetilde{H}_{k-1}^T(\eta_{1k}) & \mathbf{b}_k^T\widetilde{H}_{k-1}^T(\eta_{1k}) \\ \eta_{2k}\mathbf{b}_k^T\widetilde{H}_{k-1}^T(\eta_{2k}) & \mathbf{b}_k^T\widetilde{H}_{k-1}^T(\eta_{2k}) \\ \vdots & \\ \eta_{pk}\mathbf{b}_k^T\widetilde{H}_{k-1}^T(\eta_{pk}) & \mathbf{b}_k^T\widetilde{H}_{k-1}^T(\eta_{pk}) \end{bmatrix}, \quad \widetilde{\mathbf{e}}_p^{(k)} := \begin{bmatrix} e^{\eta_{1k}\zeta} \\ e^{\eta_{2k}\zeta} \\ \vdots \\ e^{\eta_{pk}\zeta} \end{bmatrix} \in \mathbb{R}^p. \tag{27}$$

As Theorem 3.1, we can easily derive the following result for the kth step of Problem 2.

Theorem 4.1 *For $k = 1, \ldots, m$, let the feedback vectors $\mathbf{f}_k$ and $\mathbf{g}_k$ defined by (13) with $\mathbf{w}_k$ determined by*

$$\widetilde{A}_k\mathbf{w}_k := \widetilde{G}_k \begin{bmatrix} MX_1 \\ MX_1\Lambda_1 + DX_1 \end{bmatrix} \mathbf{w}_k = \widetilde{\mathbf{e}}_p^{(k)}, \tag{28}$$

where $\widetilde{G}_k$ and $\widetilde{\mathbf{e}}_p^{(k)}$ are defined as in (27). Then $\mathbf{f}_k$ and $\mathbf{g}_k$ are real and the single-input closed-loop delayed pencil $\widetilde{P}_k(\lambda)$ in (25) has the desired eigenvalues $\{\eta_{jk}\}_{j=1}^{p}$ and the eigenpairs $\{(\lambda_j, \mathbf{x}_j)\}_{j=p+1}^{2n}$.

Notice that for any $s \in \mathbb{C}$, $H(s)$ is available from the physical tests. It follows from (26) that $\widetilde{H}_k(s)$ can be computed based on $H(s)$, $\{\mathbf{f}_i\}_{i=1}^{k}$, $\{\mathbf{g}_i\}_{i=1}^{k}$, and $e^{-s\zeta}$. In the kth step, once the feedback vectors $\{\mathbf{f}_i\}_{i=1}^{k}$ and $\{\mathbf{g}_i\}_{i=1}^{k}$ are determined, one may find an eigenvector $\mathbf{y}_{jk}$ of the single-input closed-loop delayed pencil $\widetilde{P}_k(\lambda)$ in (25) corresponding to the eigenvalue η_{jk} for $j = 1, \ldots, p$, where $(\eta_{jk}, \mathbf{y}_{jk})$ satisfies

$$\big(\eta_{jk}^2 M + \eta_{jk}(\widetilde{D}_k - e^{-\eta_{jk}\zeta}\mathbf{b}_k\mathbf{f}_k^T) + (\widetilde{K}_k - e^{-\eta_{jk}\zeta}\mathbf{b}_k\mathbf{g}_k^T)\big)\mathbf{y}_{jk} = \mathbf{0}. \tag{29}$$

Equation (29) yields

$$(\eta_{jk}^2 M + \eta_{jk}\widetilde{D}_k + \widetilde{K}_k)\mathbf{y}_{jk} = e^{-\eta_{jk}\zeta}\mathbf{b}_k(\eta_{jk}\mathbf{f}_k^T + \mathbf{g}_k^T)\mathbf{y}_{jk}.$$

Since $\tilde{\delta}_{jk} := e^{-\eta_{jk}\zeta}(\eta_{jk}\mathbf{f}_k^T + \mathbf{g}_k^T)\mathbf{y}_{jk}$ is a scalar quantity, we can find an eigenvector $\tilde{\mathbf{y}}_{jk}$ of $\widetilde{P}_k(\lambda)$ corresponding to η_{jk} by solving

$$(\eta_{jk}^2 M + \eta_{jk}\widetilde{D}_k + \widetilde{K}_k)\tilde{\mathbf{y}}_{jk} = \mathbf{b}_k, \quad \tilde{\mathbf{y}}_{jk} = \tilde{\delta}_{jk}^{-1}\mathbf{y}_{jk},$$

which gives rise to

$$\tilde{\mathbf{y}}_{jk} = \widetilde{H}_{k-1}(\eta_{jk})\mathbf{b}_k. \tag{30}$$

Especially, $\tilde{\mathbf{y}}_{jm}$ is an eigenvector of the closed-loop delayed pencil $\widetilde{P}_c(\lambda)$ in (22) corresponding to the eigenvalue μ_j for $j = 1, \ldots, p$.

Therefore, we get the following multi-step hybrid algorithm for the PQEAP with time delay.

Algorithm II: Multi-Step Hybrid Method for the PQEAP with Time Delay
Inputs:

1. The matrices $M, D, K \in \mathbb{R}^{n\times n}$, where M, D, K are symmetric with M being positive definite.
2. The control matrix $B \in \mathbb{R}^{n\times n}$ $(m \le n)$.
3. A self-conjugate set $\{\lambda_i\}_{i=1}^p$ of the spectrum of $P(\lambda)$ with associated eigenvectors $\{\mathbf{x}_j\}_{j=1}^p$.
4. A suitably chosen self-conjugate set $\{\mu_i\}_{i=1}^p$ and the measured data $\{\widetilde{H}_0(\eta_{jk}) = H(\eta_{jk}) : j = 1, \ldots, p, k = 1, \ldots, m\}$, where $\{\eta_{jk}\}$ is defined in (10).
5. ζ: time delay.

Outputs:

The real feedback matrices $F = [\mathbf{f}_1, \ldots, \mathbf{f}_m]$ and $G = [\mathbf{g}_1, \ldots, \mathbf{g}_m]$ such that the spectrum of the close-loop pencil $\widetilde{P}_c(\lambda)$ in (22) is $\{\mu_1, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n}\}$. Also, the closed-loop delayed pencil

$$\widetilde{P}_k(\lambda) = \lambda^2 M + \lambda(D - e^{-\lambda\zeta}B_k F_k^T) + (K - e^{-\lambda\zeta}B_k G_k^T)$$

has the desired eigenpairs $\{(\eta_{jk}, \tilde{\mathbf{y}}_{jk})\}_{j=1}^p$ and $\{(\lambda_j, \mathbf{x}_j)\}_{j=p+1}^{2n}$ for $k = 1, \ldots, m$, where $B_k = [\mathbf{b}_1, \ldots, \mathbf{b}_k]$, $F_k = [\mathbf{f}_1, \ldots, \mathbf{f}_k]$, and $G_k = [\mathbf{g}_1, \ldots, \mathbf{g}_k]$.

Step 1. Form the matrices Λ_1 and X_1 by (6).

Step 2. Set $\widetilde{\mathbf{e}}_p^{(1)} := [e^{\eta_{11}\zeta}, e^{\eta_{21}\zeta}, \ldots, e^{\eta_{p1}\zeta}]^T \in \mathbb{R}^p$.

Step 3. Compute $\tilde{\mathbf{y}}_{j1}$ by (30). This step needs $O(n^2 p)$ operations.

Step 4. Compute $\mathbf{w}_1$ by solving (28), which requires $O(n^2 p + np^2 + p^3)$ operations.

Step 5. Form $\mathbf{f}_1$ and $\mathbf{g}_1$ by (13). This step needs $O(n^2 + np)$ flops.

Step 6. For $k = 2, \ldots, m$

Step 6.1 For $j = 1, \ldots, p$

Step 6.1.1 For $i = 1, \dots, k-1$, compute $\widetilde{H}_i(\eta_{jk})$ by (26) in turn using $\widetilde{H}_{i-1}(\eta_{jk})$, $\mathbf{f}_i$, $\mathbf{g}_i$, and $e^{-\eta_{jk}\zeta}$. This step requires $O(n^2k)$ operations.

Step 6.1.2 Compute $\tilde{\mathbf{y}}_{jk}$ by (30). This step requires $O(n^2p)$ operations.

Step 6.2 Set $\widetilde{\mathbf{e}}_p^{(k)} := [e^{\eta_{1k}\zeta}, e^{\eta_{2k}\zeta}, \dots, e^{\eta_{pk}\zeta}]^T \in \mathbb{R}^p$.

Step 6.3 Compute $\mathbf{w}_k$ by solving (28), which needs $O(n^2p + np^2 + p^3)$ operations.

Step 6.4 Form $\mathbf{f}_k$ and $\mathbf{g}_k$ by (13). This step requires $O(n^2 + np)$ flops.

We note that Algorithm II needs $O(n^2m^2p + n^2p + np^2 + p^3)$ operations.

Finally, we give some numerical examples to illustrate the effectiveness of Algorithm II for the PQEAP with time delay.

Example 4.2 *We focus on the control system considered in Example 3.6 with time delay* $\zeta = 0.1$. *The first two eigenvalues* $\lambda_{1,2} = -0.1512 \pm 1.0372i$ *were reassigned to* $\mu_{1,2} = -0.5 \pm 1.0372i$ *and the other eigenvalues and corresponding eigenvectors were retained.*

By applying Algorithm II to Example 4.2, we obtain

$$[\mathbf{w}_1, \dots, \mathbf{w}_m] = \begin{bmatrix} -0.4453 + 0.1333i & -0.1932 + 0.1337i \\ -0.4453 - 0.1333i & -0.1932 - 0.1337i \end{bmatrix},$$

with the corresponding feedback matrices given by

$$F = \begin{bmatrix} -0.3329 & -0.1611 \\ -0.5870 & -0.2985 \\ -0.6196 & -0.3162 \end{bmatrix}, \quad G = \begin{bmatrix} -0.4810 & -0.2796 \\ 0.1355 & -0.0312 \\ 0.1503 & -0.0294 \end{bmatrix}$$

with $\|F\| = 1.0268$ and $\|G\| = 0.5784$. The errors of the close-loop eigenvalues and close-loop eigenvectors are listed as follows:

$$\begin{cases} \left\| \left(\mu_j^2 M + \mu_j (D - e^{-\mu_j \zeta} BF^T) + (K - e^{-\mu_j \zeta} BG^T) \right) \tilde{\mathbf{y}}_{jm} \right\| < 1.04 \times 10^{-14}, & 1 \le j \le p, \\ \left\| \left(\lambda_j^2 M + \lambda_j (D - e^{-\lambda_j \zeta} BF^T) + (K - e^{-\lambda_j \zeta} BG^T) \right) \mathbf{x}_j \right\| < 1.75 \times 10^{-13}, & p+1 \le j \le 2n. \end{cases}$$

Example 4.3 *We focus on the control system considered in Example 3.7 with time delay* $\zeta = 0.1$. *The first* $p = 4$ *eigenvalues with smallest absolute values are replaced by* $\mu_{2k-1,2k} = -k \pm \sqrt{-10k}$ *for* $k = 1, 2$ *and the other eigenvalues and corresponding eigenvectors were kept unchanged.*

Table 2 shows the numerical results for Example 4.3, where `tol1.` and `tol2.` denote the upper bounds for the errors of the closed-loop delayed eigenvalues and eigenvectors, i.e.,

$$\begin{cases} \left\| \left(\mu_j^2 M + \mu_j (D - e^{-\mu_j \zeta} BF^T) + (K - e^{-\mu_j \zeta} BG^T) \right) \tilde{\mathbf{y}}_{jm} \right\| < \texttt{tol1.}, & 1 \le j \le p, \\ \left\| \left(\lambda_j^2 M + \lambda_j (D - e^{-\lambda_j \zeta} BF^T) + (K - e^{-\lambda_j \zeta} BG^T) \right) \mathbf{x}_j \right\| < \texttt{tol2.}, & p+1 \le j \le 2n. \end{cases}$$

We can observe from Table 2 that the unwanted eigenvalues are reassigned to desired ones with no spillover. This agrees with our prediction.

Table 2: Numerical results for Example 4.3

n	$\|F\|$	$\|G\|$	tol1.	tol2.
10	407	659	3.65×10^{-13}	2.14×10^{-13}
20	3825	6409	2.38×10^{-12}	9.44×10^{-12}
40	39454	67907	2.63×10^{-11}	3.81×10^{-10}
80	426303	745916	2.27×10^{-10}	1.70×10^{-8}
100	922403	1619703	1.16×10^{-10}	2.14×10^{-8}

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