Minimum Norm Partial Quadratic Eigenvalue Assignment with Time Delay in Vibrating Structures Using the Receptance and the System Matrices

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Abstract

The partial quadratic eigenvalue assignment problem (PQEAP) is to compute a pair of feedback matrices such that a small number of unwanted eigenvalues in a structure are reassigned to suitable locations while keeping the remaining large number of eigenvalues and the associated eigenvectors unchanged. The problem arises in active vibration control of structures. For real-life applications, it is not enough just to compute the feedback matrices. They should be computed in such a way that both closed-loop eigenvalue sensitivity and feedback norms are as small as possible. Also, for practical effectiveness, the time-delay between the measurement of the state and implementation of the feedback controller should be considered while solving the PQEAP. These problems are usually solved using only system matrices and do not necessarily take advantage of the receptances which are available by measurements.

In this paper, we propose hybrid methods, combining the system matrices and measured receptances, for solutions of the multi-input PQEAP and the minimum-norm PQEAP, both for systems with and without time-delay. These hybrid methods are more efficient than the standard methods which only use the system matrices and no the receptances. These hybrid methods also offer several other computational advantages over the standard methods. Our results generalize the recent work by Ram, Mottershead, and Tehrani [*Linear Algebra Appl.*, 434 (2011), pp. 1689–1696]. The results of numerical experiments demonstrate the effectiveness of the proposed methods.

Keywords. Partial quadratic eigenvalue assignment, receptance measurements, time delay
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1 Introduction

Vibrating structures, such as bridges, highways, buildings, automobiles, airplanes, etc., are usually modeled by a system of second-order differential equations of the form:

$$M\ddot{\mathbf{x}}(t) + D\dot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{f}(t), \tag{1}$$

where the matrices M, D, and K are, respectively, known as the mass, damping and stiffness matrices. They are very often structured with special properties. They are symmetric and furthermore, M is usually positive definite and diagonal or tridiagonal and K is positive semi-definite and tridiagonal.

The dynamics of the system (1) are governed by the **natural frequencies** and **mode shapes**. The natural frequencies are related to the eigenvalues and mode shapes are the eigenvectors of the associated quadratic matrix pencil:

$$P(\lambda) = \lambda^2 M + \lambda D + K.$$

If each of the matrices M, D, and K is of order n, then $P(\lambda)$ has $2n$ finite eigenvalues and $2n$ associated eigenvectors under the assumption that M is nonsingular [1, 2].

One of the fundamental problems in vibration is to control the undesired vibrations, such as those caused by resonances, when vibrating structures are acted upon by some external forces, such as the wind, an earthquake, or human weight.

Resonance is caused when some natural frequencies become close or equal to the external frequencies. Therefore, mathematically, the vibration control problem is to reassign those few unwanted resonant eigenvalues to suitably chosen locations, selected by the engineers, while keeping the large number of remaining eigenvalues and their corresponding eigenvectors unchanged. The latter is known as the **no spill-over phenomenon in vibration engineering**.

In mathematics and control literature, the above problem is known as the **Partial Quadratic Eigenvalue Assignment Problem** (PQEAP). A fundamental computational challenge is to solve the PQEAP using only a small number of eigenvalues and eigenvectors of the pencil $P(\lambda)$, which are available by computation with the state-of-the-art computational techniques, such as the Jacobi-Davidson method [3] or by measurements from a vibration laboratory using limited hardware facilities.

The PQEAP as stated above is basic. For practical effectiveness, the problem must be solved by addressing several important practical issues. These include:

- **Robustness and minimum-norm feedback**: Since the eigenvalues of a matrix may be very sensitive even to small perturbations, the feedback matrices must be computed in such a way that *the closed-loop eigenvalues remain as insensitive as possible to small perturbations of the data.* Also, for applications, the feedback design should be such that the *norms of the feedback matrices are as small as possible.* These considerations lead to **robust** and **minimum-norm quadratic partial eigenvalue assignment problems**. Solutions of robust and minimum-norm problems give rise to nonlinear optimization problems. There still do not exist viable methods for numerically solving nonlinear optimization problems. Even for local minimization, the computational challenge is to be *able to compute the gradient expressions using only a few available eigenvalues and eigenvectors.*

- **Time-delay in the system**: Time-delay is an inevitable practical phenomenon. There always exists a time-delay in the application of the required control force to the structure.

  Design of feedback controllers is a much more difficult and challenging problem for a time-delay system; because it involves only $2n$ parameters whereas the closed-loop system in this case has an **infinite number** of eigenvalues. Fortunately, however, it has been shown earlier (e.g., Ram, et al. [4]) that $p < 2n$ eigenvalues can be reassigned in the time-delay case.

- **Use of receptances**: The receptance matrix corresponding to system (1) is defined by

$$H(s) = (s^2 M + sD + K)^{-1}.$$

  The entries of this matrix are available by measurements. It is, therefore, highly desirable that these measurements are used as much as possible, to ease the burden of computations of the feedback matrices.

  Some remarkable progress has been made on the solution of the PQEAP, that has addressed some of the above challenges. The PQEAP was first solved by Datta, Elhay, and Ram [5] in the single-input case and later their method was generalized to the multi-input case by Datta and Sarkissian [6] and by Ram and Elhay [7] and Sarkissian [8].

  Robust and minimum-norm problems have been considered by Bai, Datta, and Wang [9], Brahma and Datta [10], Chan, Lam, and Ho [11], Chu and Datta [12], Lam and Tam [13, 14], Lam and Yan [15], and Qian and Xu [16], Datta [1] and the references therein, etc. Meanwhile, Mottershead, Tehrani and Ram [17] and Ram and Mottershead [18] studied several aspects of active vibration control using receptance measurements. Recently, Ram, Mottershead, and Tehrani [19] proposed a hybrid method, combining receptances and system matrices, to solve the single-input quadratic eigenvalue assignment and extended their method to the time-delay case. An important observation made in the paper is that the *quadratic partial eigenvalue assignment problem in the time-delay case can not be solved by using receptance alone*-a hybrid approach is needed.

In this paper, we

- **First**, generalize the single-input hybrid method of Ram, Mottershead, and Tehrani [19] to the solution of the multi-input PQEAP.

- **Then**, propose a new optimization-based hybrid method for computing minimum feedback norms of the multi-input PQEAP, for both with and without time-delay.

  The proposed hybrid method offers several computational advantages over the standard methods (without the use of the receptances) that were proposed earlier for the PQEAP (Datta, et al [9, 10, 20], [5], [12], [21], etc.)

  - **First**, the need to solve the Sylvester-matrix equation in computation of the feedback matrices is eliminated.

– **Second**, the eigenvectors of the closed-loop pencil corresponding to the eigenvalues that are to be reassigned are not needed in this hybrid method. They are readily available from the entries of the receptance matrix (see Equation (15)).

- **More importantly**, the new hybrid method does not involve computation of the parametric matrix. The proper choice of this parametric matrix for the methods in [9, 10, 20] is crucial–it needs to be chosen in such a way that the solution of the associated Sylvester equation becomes nonsingular [See Equation (5)]. At this time, there is no systematic way to choose this matrix, except by trial-and-error processes (see Remark 3.2 in Section 3).

Results of numerical experiments show that in all cases, the hybrid method was quite effective: (i) the eigenvalues are reassigned quite accurately, (ii) no spill-over is nicely maintained, and (iii) feedback norms are considerably smaller with the hybrid methods than those obtained without the use of receptances.

2 Problem Statements

Suppose a control force of the form $\mathbf{f}(t) = B\mathbf{u}(t)$, where B is the control matrix of order $n \times m$ ($m \leq n$), is applied to the structure model by (1). Choosing

$$\mathbf{u}(t) = F^T\dot{\mathbf{x}}(t) + G^T\mathbf{x}(t), \tag{2}$$

where F and G are two $n \times m$ feedback matrices, we have the closed-loop control system:

$$M\ddot{\mathbf{x}}(t) + (D - BF^T)\dot{\mathbf{x}}(t) + (K - BG^T)\mathbf{x}(t) = \mathbf{0}. \tag{3}$$

The dynamics of this closed-loop system are now governed by the eigenvalues and eigenvectors of the closed-loop quadratic pencil

$$P_c(\lambda) = \lambda^2 M + \lambda(D - BF^T) + (K - BG^T).$$

Let $\{\lambda_1, \ldots, \lambda_p; \lambda_{p+1}, \ldots, \lambda_{2n}\}$ be the spectrum of $P(\lambda)$ with associated eigenvectors $\{\mathbf{x}_1, \ldots, \mathbf{x}_p; \mathbf{x}_{p+1}, \ldots, \mathbf{x}_{2n}\}$. Assume that the eigenvalues $\{\lambda_i\}_{i=1}^p$ ($p \ll 2n$) have been identified as resonant and the remaining $2n-p$ eigenvalues $\{\lambda_{p+1}, \ldots, \lambda_{2n}\}$ are acceptable. Suppose that $\{\mu_i\}_{i=1}^p$ are suitably chosen numbers.

2.0.1 The Partial Quadratic Eigenvalue Assignment Problem (PQEAP)

Find the feedback matrices $F \in \mathbb{R}^{n\times m}$ and $G \in \mathbb{R}^{n\times m}$ such that the spectrum of the closed-loop pencil $P_c(\lambda)$ is $\{\mu_1, \ldots, \mu_p; \lambda_{p+1}, \ldots, \lambda_{2n}\}$ and the eigenvectors $\{\mathbf{x}_{p+1}, \ldots, \mathbf{x}_{2n}\}$ corresponding to the eigenvalues $\{\lambda_{p+1}, \ldots, \lambda_{2n}\}$ remain unchanged.

2.0.2 Minimum-Norm and Robust QPEAP

For practical effectiveness, an active vibration design scheme must take into consideration the robustness aspect of the design due to small variations of the data. To ensure robustness in the design, the feedback matrices should be such that (i) they have norms as small as possible, and (ii) the closed-loop eigenvector matrices be well-conditioned.

Both these problems are intertwined (see Datta [1]). However, we only consider the minimum-norm feedback problem in this paper. This is practically acceptable since small feedback gains yield smaller control signals and thus reduce the energy consumption. Moreover, small feedback gains is useful to the reduction of noise amplification.

Minimum-norm QPEAP: The feedback matrices F and G should be computed in such a way that in addition to solving the basic PQEAP, their norms become as small as possible. That is,

$$I = \frac{1}{2}(\|F\|_F^2 + \|G\|_F^2)$$

is minimized. Here, $\|\cdot\|_F$ denotes the Frobenius matrix norm.

Remark 2.1 *A more challenging problem is to reduce the feedback norm and the sensitivity of closed-loop eigenvalues simultaneously. As in [9], it seems that a natural choice is to minimize*

$$\frac{\alpha}{2}(\|W\|_F^2 + \|W^{-1}\|_F^2) + \frac{1-\alpha}{2}(\|F\|_F^2 + \|G\|_F^2),$$

where $0 \leq \alpha \leq 1$ *is a weighting parameter and* $J := \|W\|_F^2 + \|W^{-1}\|_F^2$ *can be seen as a possible measure of the sensitivity of closed-loop eigenvalues with*

$$W = \begin{bmatrix} \mathbf{z}_1 & \cdots & \mathbf{z}_p & \mathbf{x}_{p+1} & \cdots & \mathbf{x}_{2n} \\ \mu_1\mathbf{z}_1 & \cdots & \mu_p\mathbf{z}_p & \lambda_{p+1}\mathbf{x}_{p+1} & \cdots & \lambda_{2n}\mathbf{x}_{2n} \end{bmatrix}.$$

Here, $\mathbf{z}_j$ *is an eigenvector of the closed-loop pencil* $P_c(\lambda)$ *corresponding to the new eigenvalue* μ_j *for* $j = 1, \ldots, p$. *An interesting topic would be to develop a hybrid method using the receptance and the system matrices* M, D, K, *which needs further investigation.*

3 Solution of the PQEAP without the Use of Receptances

In what follows, we assume that M, D and K are real symmetric with M positive definite. Let $\|\cdot\|$ and $\|\cdot\|_2$ denote the Euclidean vector norm and the matrix 2-norm, respectively. Denote by $A(:,k)$ and $A(k,:)$ the kth column and the kth row of a matrix A, respectively. I_n is the identity matrix of order n. Suppose that (i) $\{\mu_1, \ldots, \mu_p\} \cap \{\lambda_1, \ldots, \lambda_{2n}\} = \emptyset$ and $\{\lambda_1, \ldots, \lambda_p\} \cap \{\lambda_{p+1}, \ldots, \lambda_{2n}\} = \emptyset$, (ii) the control matrix B has full column rank, and (iii) $(P(\lambda), B)$ is partially controllable with respect to the eigenvalues $\lambda_1, \ldots, \lambda_p$, i.e.,

$$\text{rank } (P(\lambda_i), B) = n, \ i = 1, \ldots, p.$$

Let

$$\Lambda_1 = \text{diag}(\lambda_1, \ldots, \lambda_p), \quad \Lambda_2 = \text{diag}(\lambda_{p+1}, \ldots, \lambda_{2n})$$

and

$$X_1 = [\mathbf{x}_1, \ldots, \mathbf{x}_p], \quad X_2 = [\mathbf{x}_{p+1}, \ldots, \mathbf{x}_{2n}].$$

Bai, Datta, and Wang [9] proved the following result on the solution of the basic PQEAP. A similar result also appears in Brahma and Datta [20].

Lemma 3.1 *Given a self-conjugate set of p complex numbers* $\{\mu_k\}_{k=1}^p$.

(a) ***(No spill-over part)*** *For arbitrary* $\Phi \in \mathbb{C}^{m\times p}$*, the feedback matrices* F *and* G *given by*

$$F = MX_1\Phi^T \quad \textit{and} \quad G = (MX_1\Lambda_1 + DX_1)\Phi^T \tag{4}$$

are such that

$$MX_2\Lambda_2^2 + (D - BF^T)X_2\Lambda_2 + (K - BG^T)X_2 = 0.$$

That is, the $2n - p$ *eigenvalues which are not reassigned and the associated eigenvectors remain preserved.*

(b) ***(Eigenvalue assignment part)*** *Choose* $\Phi \in \mathbb{C}^{m\times p}$ *such that* $\Phi Z = \Gamma$*, where* $\Gamma = [\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_p] \in \mathbb{C}^{m\times p}$ *is an arbitrary nonzero matrix such that if* $\mu_j = \bar{\mu}_k$*, then* $\boldsymbol{\gamma}_j = \bar{\boldsymbol{\gamma}}_k$*, and* Z *is the solution to the Sylvester equation*

$$\Lambda_1 Z - Z\Sigma = -X_1^T B\Gamma, \tag{5}$$

where $\Sigma = \mathrm{diag}(\mu_1, \ldots, \mu_p)$*. Then, the feedback matrices* F *and* G *defined by* (4) *are* ***real*** *and the p given numbers* $\{\mu_1, \ldots, \mu_p\}$ *become a part of the spectrum of the closed-loop pencil* $P_c(\lambda)$.

(c) ***(Explicit solution)*** *Suppose that* Z *is nonsingular. Let* $C = [\Lambda_1 X_1^T M + X_1^T D, X_1^T M]$*. Then* $[G^T, F^T] = \Gamma Z^{-1} C$.

Remark 3.2 ***Non-Uniqueness of the Solution*** *(i) Since it is possible for* (5) *to be satisfied for many choices of* Γ*, it follows that the solution to the PQEAP is not unique.*

Nonsingularity of the matrix Z *(ii) If an initial choice of* Γ *does not yield a nonsingular solution* Z *of* (5)*, a different* Γ *has to be chosen and the process is repeated until a nonsingular solution is obtained. (Notice that a nonsingular* Z *will guarantee a solution* Φ *of the algebraic system:* $\Phi Z = \Gamma$ *in part (b) of Lemma 3.1).*

4 Partial Quadratic Eigenvalue Assignment Using the Partial Measured Receptance and the System Matrices

In this section, we propose a hybrid method for solving the PQEAP that make use of both receptance measurements and the system matrices M, D, K. For any $s \in \mathbb{C}$, the receptance matrix $H(s)$ to the open-loop pencil $P(\lambda)$ is defined by

$$H(s) = (s^2M + sD + K)^{-1},$$

which can be measured without any explicit knowledge of the matrices M, D, K [22]. Let $H_c(s)$ denote the receptance matrix corresponding to the closed-loop pencil $P_c(\lambda)$, i.e.,

$$H_c(s) = \big(s^2M + s(D - BF^T) + (K - BG^T)\big)^{-1} \quad \forall s \in \mathbb{C}.$$

By the Sherman-Morrison-Woodbury formula [23], we have

$$H_c(s) = H(s) + H(s)B\big(I_m - (G + sF)^T H(s)B\big)^{-1}(G + sF)^T H(s). \tag{6}$$

Notice that $\det(H_c(\mu_j)) \to \infty$ for $j = 1, \ldots, p$. It follows from (6) that

$$\det\big(I_m - (G + \mu_j F)^T H(\mu_j)B\big) = 0, \quad j = 1, \ldots, p, \tag{7}$$

i.e.,

$$\det\left(\left[\mu_j B^T H(\mu_j),\ B^T H(\mu_j)\right]\begin{bmatrix} F \\ G \end{bmatrix} - I_m\right) = 0, \quad j = 1, \ldots, p.$$

From the above observations, the following result on the solvability of the PQEAP follows immediately:

Theorem 4.1 ***(Hybrid Solution of PQEAP)****: Given $B \in \mathbb{R}^{n\times m}$, Λ_1, X_1, and the set of p self-conjugate numbers $\{\mu_j\}_{j=1}^p$. Let $\Phi \in \mathbb{C}^{m\times p}$ be any matrix satisfying*

$$\det\left(\left[\mu_j B^T H(\mu_j),\ B^T H(\mu_j)\right]\begin{bmatrix} MX_1 \\ MX_1\Lambda_1 + DX_1 \end{bmatrix}\Phi^T - I_m\right) = 0, \quad j = 1, \ldots, p. \tag{8}$$

Then the feedback matrices F and G defined by (4) *with the matrix Φ defined by above* (8) *solve the PQEAP.*

Proof: No Spill-over Part (i) By Lemma 3.1 (a), we know that for an arbitrary $\Phi \in \mathbb{C}^{m\times p}$, the feedback matrices F and G defined by (4) are such that the closed-loop pencil $P_c(\lambda)$ has the $2n - p$ eigenpairs $\{(\lambda_j, \mathbf{x}_j)\}_{j=p+1}^{2n}$.

Eigenvalue Assignment Part (ii) Sylvester's determinant theorem [24] states that if C_1, C_2 are matrices of size n_1-by-n_2 and n_2-by-n_1, respectively, then

$$\det(I_{n_1} + C_1C_2) = \det(I_{n_2} + C_2C_1). \tag{9}$$

Thus, for any nonsingular n_1-by-n_1 matrix C_3,

$$\det(C_3 + C_1C_2) = \det(C_3)\det(I_{n_2} + C_2C_3^{-1}C_1). \tag{10}$$

This, together with (8), (4), and (7), leads to:

$$\begin{aligned}
\det(P_c(\mu_j)) &= \det\Big(\mu_j^2 M + \mu_j(D - BF^T) + (K - BG^T)\Big) \\
&= \det\Big((\mu_j^2 M + \mu_j D + K) - B(G + \mu_j F)^T\Big) \\
&= \det\Big(\mu_j^2 M + \mu_j D + K\Big)\det\big(I_m - (G + \mu_j F)^T H(\mu_j)B\big) \\
&= 0, \quad j = 1, \ldots, p.
\end{aligned}$$

Therefore, the closed-loop pencil $P_c(\lambda)$ contains the eigenvalues $\{\mu_j\}_{j=1}^p$. □

Remark 4.2 ***Recovering of Ram-Mottershead-Tehrani Result*** *We point out that Theorem* 4.1 *generalizes the recent work by Ram, Mottershead, and Tehrani [19] in the sense that they presented a method for the single-input case only where the solution is unique whereas Theorem* 4.1 *provides a hybrid method for the multi-input case where the PQEAP has multiple solutions depending on different choice of the parameter* Φ *satisfying the conditions (8). In particular, when* $m = 1$, *i.e.,* $F = \mathbf{f} \in \mathbb{R}^n$, $G = \mathbf{g} \in \mathbb{R}^n$, $B = \mathbf{b} \in \mathbb{R}^n$, *and* $\Phi = \boldsymbol{\phi}^T \in \mathbb{C}^{1\times p}$, *it is easy to check that* $\mathbf{f} = MX_1\boldsymbol{\phi}$ *and* $\mathbf{g} = (MX_1\Lambda_1 + DX_1)\boldsymbol{\phi}$, *where* $\boldsymbol{\phi}$ *is determined by*

$$\boldsymbol{\phi} = \left(\Psi \begin{bmatrix} MX_1 \\ MX_1\Lambda_1 + DX_1 \end{bmatrix}\right)^{-1} \mathbf{p}$$

with

$$\Psi = \begin{bmatrix} \mu_1\mathbf{b}^T H(\mu_1) & \mathbf{b}^T H(\mu_1) \\ \vdots & \vdots \\ \mu_p\mathbf{b}^T H(\mu_p) & \mathbf{b}^T H(\mu_p) \end{bmatrix} \quad \text{and} \quad \mathbf{p} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^p.$$

The above result is similar to the one proved by Ram et. al. [19] for the single-input case as in [19, Lemma 3].

Remark 4.3 *We see from the proof of Theorem* 4.1 *that the requirements in* (8) *are equivalent to*

$$\det(P_c(\mu_j)) = 0, \quad j = 1, \ldots, p.$$

We also remark that if the matrix $\Phi = [\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_p] \in \mathbb{C}^{m\times p}$ *determined by* (8) *is such that* $\boldsymbol{\phi}_j = \bar{\boldsymbol{\phi}}_k$ *if* $\lambda_j = \bar{\lambda}_k$, *then the feedback matrices* F *and* G *defined in* (4) *are real (see the numerical results below).*

4.1 Hybrid Computation of Minimum-Norm Feedback

From Theorem 4.1, it is obvious that $\Phi \in \mathbb{C}^{m\times p}$ satisfying the condition (8) is not unique. Therefore, there exist many solutions to the PQEAP. To reduce the energy consumption and signal noises, it is important that the norms of the feedback matrices F and G are as small as possible. The minimum-norm feedback problem may be formulated (in hybrid sense) as:

$$\begin{array}{ll} \min & f(Y) := \frac{1}{2}\|MX_1Y\|_F^2 + \frac{1}{2}\|(MX_1\Lambda_1 + DX_1)Y\|_F^2 \\ \text{s.t.} & \mathbf{g}(Y) = \mathbf{0}, \end{array} \tag{11}$$

where $\mathbf{g}(Y) = (g_1(Y), \ldots, g_p(Y))^T \in \mathbb{C}^p$ with $g_j(Y) : \mathbb{C}^{p\times m} \to \mathbb{C}$ defined by

$$g_j(Y) := \det\left([\mu_j B^T H(\mu_j),\, B^T H(\mu_j)] \begin{bmatrix} MX_1 \\ MX_1\Lambda_1 + DX_1 \end{bmatrix} Y - I_m\right).$$

We note that if Y^* is the solution to Problem (11), then the minimum norm solution to the PQEAP is given by

$$F = MX_1(\Phi^*)^T \quad \text{and} \quad G = (MX_1\Lambda_1 + DX_1)(\Phi^*)^T,$$

where $\Phi^* = (Y^*)^T$.

Now, we present an optimization method to solve Problem (11). Solving Problem (11) is equivalent to finding $Y \in \mathbb{C}^{p\times m}$ and $\boldsymbol{\xi} \in \mathbb{C}^p$ such that

$$\begin{cases} \nabla f(Y) + \boldsymbol{\xi}^T \nabla \mathbf{g}(Y) = \mathbf{0}, \\ \mathbf{g}(Y) = \mathbf{0} \end{cases}$$

or

$$F(Y, \boldsymbol{\xi}) := \begin{bmatrix} \nabla f(Y) + \boldsymbol{\xi}^T \nabla \mathbf{g}(Y) \\ \mathbf{g}(Y) \end{bmatrix} = \mathbf{0}. \tag{12}$$

Notice that

$$\nabla f(Y) = \begin{bmatrix} MX_1 \\ MX_1\Lambda_1 + DX_1 \end{bmatrix}^H \begin{bmatrix} MX_1 \\ MX_1\Lambda_1 + DX_1 \end{bmatrix} Y$$

and for $j = 1, \ldots, p$,

$$\nabla g_j(Y) = A_j^T \mathrm{adj}(A_j Y - I_m)^T, \quad A_j := \left[\mu_j B^T H(\mu_j),\, B^T H(\mu_j)\right] \begin{bmatrix} MX_1 \\ MX_1\Lambda_1 + DX_1 \end{bmatrix},$$

where $\mathrm{adj}(\cdot)$ means the adjoint of a square matrix. The nonlinear equations in (12) can be solved by the classical Gauss-Newton, Levenberg-Marquardt, or trust-region-reflective methods [25, 26, 27]. In the following, we show how the closed-loop eigenvectors corresponding to the eigenvalues $\mu_1, \ldots, \mu_p$ can be computed using the measured receptances only.

Characterization of the Eigenvectors Once the minimum norm solution is available, one may compute an eigenvector $\mathbf{z}_j$ of the closed-loop pencil $P_c(\lambda)$ corresponding to the new eigenvalue μ_j for $j = 1, \ldots, p$, where $(\mu_j, \mathbf{z}_j)$ satisfies

$$\left(\mu_j^2 M + \mu_j (D - BF^T) + (K - BG^T)\right) \mathbf{z}_j = \mathbf{0}. \tag{13}$$

Let $B = [\mathbf{b}_1, \ldots, \mathbf{b}_m]$, $F = [\mathbf{f}_1, \ldots, \mathbf{f}_m]$, and $G = [\mathbf{g}_1, \ldots, \mathbf{g}_m]$. Define

$$D_k := D - \sum_{i=1}^{k} \mathbf{b}_i \mathbf{f}_i^T \quad \text{and} \quad K_k := K - \sum_{i=1}^{k} \mathbf{b}_i \mathbf{g}_i^T, \quad k = 1, \ldots, m,$$

where $D_0 := D$ and $K_0 := K$. For any $s \in \mathbb{C}$, the receptance matrices $\{H_k(s)\}$ are

$$H_k(s) := (s^2 M + s D_k + K_k)^{-1}, \quad k = 0, 1, \ldots, m,$$

where $H_0(s) = H(s)$. By the Sherman-Morrison formula [28, 29], we get

$$H_k(s) = H_{k-1}(s) + \frac{H_{k-1}(s)\mathbf{b}_k (\mathbf{g}_k^T + s\mathbf{f}_k^T) H_{k-1}(s)}{1 - (\mathbf{g}_k^T + s\mathbf{f}_k^T) H_{k-1}(s) \mathbf{b}_k}, \quad k = 1, \ldots, m. \tag{14}$$

We observe from (14) that the matrices $\{H_k(s)\}$ can be computed recursively given $\{\mathbf{f}_i\}_{i=1}^k$, and $\{\mathbf{g}_i\}_{i=1}^k$.

Suppose that there exists an index w such that $\mathbf{f}_w \neq \mathbf{0}$ or $\mathbf{g}_w \neq \mathbf{0}$ while $\mathbf{f}_l = \mathbf{0}$ and $\mathbf{g}_l = \mathbf{0}$ for all $l = w+1, \ldots, m$. Then (13) reduces to

$$\left(\mu_j^2 M + \mu_j D_{w-1} + K_{w-1}\right) \mathbf{z}_j = \mathbf{b}_w(\mu_j \mathbf{f}_w^T + \mathbf{g}_w^T)\mathbf{z}_j.$$

In this case, $\delta_{jw} := (\mu_j \mathbf{f}_w^T + \mathbf{g}_w^T)\mathbf{z}_j$ is a nonzero scalar quantity. Defining now $\hat{\mathbf{z}}_j := \delta_{jw}^{-1}\mathbf{z}_j$, we get

$$\left(\mu_j^2 M + \mu_j D_{w-1} + K_{w-1}\right) \hat{\mathbf{z}}_j = \mathbf{b}_w.$$

Thus finding an eigenvector $\hat{\mathbf{z}}_j$ of $P_c(\lambda)$ corresponding to the eigenvalue μ_j is equivalent to computing:

$$\hat{\mathbf{z}}_j = H_{w-1}(\mu_j)\mathbf{b}_w. \tag{15}$$

This shows that *once the quantities $H(\mu_j)$ are available from measurements, the eigenvectors $\{\hat{\mathbf{z}}_j\}$ are readily computed from them.*

Algorithm 4.1 (Hybrid Algorithm for Norm Minimization without Time Delay):
Inputs:

1. The matrices $M, D \in \mathbb{R}^{n\times n}$, where $M^T = M > 0$ and $D^T = D$.
2. The control matrix $B \in \mathbb{R}^{n\times m}$ $(m \leq n)$.
3. A self-conjugate subset $\{\lambda_j\}_{j=1}^p$ of the spectrum of $P(\lambda)$ and the associated eigenvectors $\{\mathbf{x}_j\}_{j=1}^p$.
4. A suitably chosen self-conjugate set $\{\mu_j\}_{j=1}^p$ and the measured receptances $\{H(\mu_j)\}_{j=1}^p$.
5. $\epsilon =$ A tolerance for gradient.

Outputs:

(i) The real feedback matrices F and G such that the spectrum of the close-loop pencil (4) is the set $\{\mu_1, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n}\}$ and the objective function $f(Y)$ defined in Problem (11) is minimized.

(ii) The closed-loop eigenvectors $\{\hat{\mathbf{z}}_j\}$ corresponding to the eigenvalues $\mu_1, \ldots, \mu_p$.

Step 1. Form the matrices Λ_1 and X_1 from the given eigenvalues and eigenvectors.

Step 2. Compute the solution Y^* to Problem (11) by solving (12). This is done by using the MATLAB function `fsolve` with the termination tolerance ϵ on the function value. This step requires $O(n^2p + m^3p^3)$ flops.

Step 3. Form the feedback matrices $F = [\mathbf{f}_1, \ldots, \mathbf{f}_m] = MX_1(\Phi^*)^T$ and $G = [\mathbf{g}_1, \ldots, \mathbf{g}_m] = (MX_1\Lambda_1 + DX_1)(\Phi^*)^T$, where $\Phi^* = (Y^*)^T$. This step needs $O(n^2p)$ operations.

Step 4. For $k = m, m-1, \ldots, 1$, determine the index w such that $\mathbf{f}_w \neq \mathbf{0}$ or $\mathbf{g}_w \neq \mathbf{0}$ while $\mathbf{f}_l = \mathbf{0}$ and $\mathbf{g}_l = \mathbf{0}$ for all $l = w+1, \ldots, m$.

Step 5. For $j = 1, \ldots, p$, do

Step 5.1 Compute $H_{w-1}(\mu_j)$ successively by (14) using $H(\mu_j)$, $\{\mathbf{f}_i\}_{i=1}^{w-1}$, and $\{\mathbf{g}_i\}_{i=1}^{w-1}$. This step needs $O(n^2w)$ operations.

Step 5.2 Compute $\hat{\mathbf{z}}_j$ by (15). This step needs $O(n^2)$ operations.

4.1.1 Computational Advantages of Algorithm 4.1

The total computational complexity for Algorithm 4.1 is $O(n^2p + n^2mp + m^3p^3)$ operations. As stated earlier, our proposed optimization method has *some advantages over the methods* in [9, 10, 20]. **First**, this method avoids solving Sylvester equations. **Second**, the initial guess for the parameter Y (i.e., Φ) can be chosen arbitrarily while in the methods in [9, 10, 20], as shown in Lemma 3.1, one must choose the parameter $\Gamma = [\boldsymbol{\gamma}_1, \ldots, \boldsymbol{\gamma}_p] \in \mathbb{C}^{m\times p}$ such that if $\mu_j = \bar{\mu}_k$, then $\boldsymbol{\gamma}_j = \bar{\boldsymbol{\gamma}}_k$. Furthermore, the solution Z to the Sylvester equation (5) is not guaranteed to be nonsingular. The matrix Γ has to be chosen in a trial-and-error basis until Z is nonsingular. **Third**, computing the eigenvectors corresponding the new eigenvalues $\{\mu_j\}_{j=1}^p$ needs $O(n^2mp)$ operations, which is much smaller than $O(n^3p)$ operations required by the methods in [9, 10, 20], since $m, p \ll n$.

4.1.2 Closed-Loop Condition Estimation

Assume that the closed-loop matrix

$$\widehat{A} := \begin{bmatrix} O & I_n \\ -M^{-1}(K - BG^T) & -M^{-1}(D - BF^T) \end{bmatrix}$$

is diagonalizable. Then the smallest spectral condition number of $\widehat{A}$ is given by [30]

$$\kappa_{2S} := \min_{Q\in\mathcal{K}} \|Q\|_2 \|Q^{-1}\|_2,$$

where $\mathcal{K} := \{Q \in \mathbb{C}^{2n\times 2n} : Q^{-1}\widehat{A}Q = \mathrm{diag}(\mu_1, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n})\}$. Since

$$Q_0 := \begin{bmatrix} \widehat{\mathbf{z}}_1 & \cdots & \widehat{\mathbf{z}}_p & \mathbf{x}_{p+1} & \cdots & \mathbf{x}_{2n} \\ \mu_1\widehat{\mathbf{z}}_1 & \cdots & \mu_p\widehat{\mathbf{z}}_1 & \lambda_{p+1}\mathbf{x}_{p+1} & \cdots & \lambda_{2n}\mathbf{x}_{2n} \end{bmatrix}$$

is an eigenvector matrix of $\widehat{A}$, κ_{2S} can be expressed by

$$\kappa_{2S} = \min_{\mathbf{v}\in\mathbb{R}^{2n-1}} \|Q_0 \mathrm{diag}(\mathbf{v}, 1)\|_2 \|\mathrm{diag}(\mathbf{v}, 1)^{-1} Q_0^{-1}\|_2.$$

In addition, the condition number $\kappa_{2N} := \|\overline{Q}\|_2 \|\overline{Q}^{-1}\|_2$, where the columns of $\overline{Q}$ are those of Q_0, with unity 2-norm, gives an estimate of κ_{2S}: $\kappa_{2N}/\sqrt{2n} \le \kappa_{2S} \le \kappa_{2N}$ [31].

4.1.3 Results of Numerical Experiment

In the following, we present results of numerical experiments to illustrate the effectiveness of the proposed method. In our numerical tests, we set the tolerance for gradient to be $\epsilon = 1.0 \times 10^{-6}$. The numerical tests were implemented in `MATLAB 7.10` and run on a PC Intel Pentium IV of 3.00 GHZ CPU.

Example 4.4 *Consider the second-order control system* (3) *with* $n = 3$ *and* $m = 2$. *Here,*

$$M = 10I_3, \quad D = 0, \quad K = \begin{bmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{bmatrix}.$$

The open-loop eigenvalues are: $\{\pm 3.6039\text{i}, \pm 2.4940\text{i}, \pm 0.8901\text{i}\}$*. The first two eigenvalues* $\{\pm 3.6039\text{i}\}$ *are reassigned to* $\{-1, -2\}$*, the other eigenvalues are kept unchanged.*

By using Algorithm 4.1 to Example 4.4, we obtain

$$\Phi^* = \begin{bmatrix} -0.0697 + 0.0686\text{i} & -0.0697 - 0.0686\text{i} \\ 0.9811 - 0.9653\text{i} & 0.9811 + 0.9653\text{i} \end{bmatrix}$$

and the minimum norm feedback matrices are given as follows:

$$F = \begin{bmatrix} 0.4502 & -6.3323 \\ -1.0116 & 14.2285 \\ 0.8112 & -11.4104 \end{bmatrix}, \quad G = \begin{bmatrix} -1.6489 & 23.1929 \\ 3.7050 & -52.1140 \\ -2.9712 & 41.7922 \end{bmatrix},$$

where

$$\|F\|_F = 19.3554, \quad \|G\|_F = 70.8918,$$
$$\kappa_{2N} = 127.7192, \quad \kappa_{2S} = 120.7465.$$

The close-loop eigenvalues and associated eigenvectors satisfy:

$$\begin{cases} \left|\det\left(\mu_j^2 M + \mu_j(D - BF^T) + (K - BG^T)\right)\right| < 5.3 \times 10^{-10}, & 1 \le j \le p, \\ \left\|\left(\mu_j^2 M + \mu_j(D - BF^T) + (K - BG^T)\right)\hat{\mathbf{z}}_j\right\| < 1.2 \times 10^{-14}, & 1 \le j \le p, \\ \left\|\left(\lambda_j^2 M + \lambda_j(D - BF^T) + (K - BG^T)\right)\mathbf{x}_j\right\| < 8.1 \times 10^{-14}, & p+1 \le j \le 2n. \end{cases}$$

Thus,

(i) the two eigenvalues are assigned correctly and

(ii) the remaining eigenvalues and eigenvectors were computationally kept unchanged.

(iii) The condition numbers κ_{2N} and κ_{2S} are accurately estimated.

Example 4.5 *[9, 10] Consider the second-order control system* (3) *with* $m = 2$ *and* $n = 10$*,* $50, 100, 200, 400$*, where*

$$M = 4I_n, \quad D = 4I_n \quad K = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & \cdots & 0 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

The first $p = 2$ *eigenvalues with smallest absolute values are replaced by* $\{-0.1, \; -0.2\}$ *and the other eigenvalues are kept unchanged.*

Here, we compare the feedback norms obtained by the method based on Theorem 4.1 (*without norm-minimization*) with those by Algorithm 4.1 (**with norm-minimization**). For the method based on Theorem 4.1, two different choices of Φ were made and the computed feedback norms were identical.

4.1.4 Results on Eigenvalue and Eigenvector Assignment

$$\begin{cases} \left|\det\left(\mu_j^2 M + \mu_j(D - BF^T) + (K - BG^T)\right)\right| < \texttt{tol1.}, & 1 \le j \le p, \\ \left\|\left(\mu_j^2 M + \mu_j(D - BF^T) + (K - BG^T)\right)\hat{\mathbf{z}}_j\right\| < \texttt{tol2.}, & 1 \le j \le p, \\ \left\|\left(\lambda_j^2 M + \lambda_j(D - BF^T) + (K - BG^T)\right)\mathbf{x}_j\right\| < \texttt{tol3.}, & p+1 \le j \le 2n. \end{cases}$$

where `tol1`, `tol2`, and `tol3` are computed upper bounds.

Table 1: **Comparison of Feedback Norms with and without Minimization**

	Feedback Norms without Minimization		Feedback Norm with Minimization (Alg. 4.1)				
n	$\|F\|_F$	$\|G\|_F$	$\|F\|_F$	$\|G\|_F$	`tol1.`	`tol2.`	`tol3.`
10	11.1312	11.0814	1.4333	1.4114	1.2×10^{-12}	5.9×10^{-12}	2.0×10^{-14}
50	698	698	3.4538	3.4516	5.3×10^{-15}	1.4×10^{-13}	3.6×10^{-14}
100	3965	3965	4.8954	4.8945	1.6×10^{-12}	1.3×10^{-11}	1.6×10^{-12}
200	22456	22455	6.9269	6.9266	5.6×10^{-14}	2.4×10^{-13}	2.5×10^{-12}
400	127060	127059	9.7975	9.7974	7.7×10^{-12}	5.8×10^{-11}	7.6×10^{-12}

Observations:

- The two eigenvalues were accurately assigned.
- The remaining eigenvalues and eigenvectors remain invariant numerically.
- The feedback norms with norm-minimization by Algorithm 4.1 were considerably smaller than those without norm-minimization.

4.1.5 Comparison of System Responses

To further illustrate the effectiveness of Algorithm 4.1, we compare system responses for open-loop and closed-loop systems without norm minimization. We also compare system responses for open-loop and closed-loop systems with norm minimization under different small perturbations of the stiffness matrix K.

Example 4.6 *Consider the second-order control system* (3) *with* $n = 3$ *and* $m = 2$*, where*

$$M = 2I_3, \quad D = \begin{bmatrix} 2.5 & 2.0 & 0 \\ 2.0 & 1.7 & 0.4 \\ 0 & 0.4 & 2.5 \end{bmatrix}, \quad K = \begin{bmatrix} 20 & 16 & 0 \\ 16 & 17 & 5 \\ 0 & 5 & 25 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{bmatrix}.$$

Then open-loop eigenvalues are $\{-1.0303 \pm 4.0868\mathrm{i}, -0.6365 \pm 3.4475\mathrm{i}, -0.0082 \pm 0.9571\mathrm{i}\}$. *We replace the two open-loop complex eigenvalues* $\{-0.0082 \pm 0.9571\mathrm{i}\}$ *by* $\{-0.5 \pm 0.9571\mathrm{i}\}$. *The other eigenvalues are kept unchanged.*

For simplicity, we choose $\Phi \in \mathbb{C}^{m\times p}$ such that

$$A_j(m,:)\Phi^T := \left[\mu_j B(:,m)^T H(\mu_j),\, B(:,m)^T H(\mu_j)\right] \begin{bmatrix} MX_1 \\ MX_1\Lambda_1 + DX_1 \end{bmatrix} \Phi^T = I(m,:)$$

for $j = 1, \ldots, p$, i.e.,

$$\Phi = \left(\begin{bmatrix} A_1(m,:) \\ A_2(m,:) \\ \vdots \\ A_p(m,:) \end{bmatrix}^{-1} \begin{bmatrix} I(m,:) \\ I(m,:) \\ \vdots \\ I(m,:) \end{bmatrix} \right)^T .$$

By Theorem 4.1, we obtain the feedback matrices

$$F = \begin{bmatrix} 0 & -2.8177 \\ 0 & 3.2096 \\ 0 & -0.6878 \end{bmatrix} \quad \text{and} \quad G = \begin{bmatrix} 0 & -0.8301 \\ 0 & 0.3300 \\ 0 & -0.3544 \end{bmatrix}$$

with $\|F\|_F = 4.3260$ and $\|G\|_F = 0.9610$. Of course, one may replace m by k with $1 \le k \le m$.

We now compare the system responses of open-loop and closed-loop systems with the feedback matrices F and G obtained by Theorem 4.1. Figure 1 depicts the base 10 logarithm of the norm of the system responses over the given time period. The initial condition is $\mathbf{w}(0) = 0.01 \cdot \mathbf{1}_{2n}$, where

$$\mathbf{w}(t) = \begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} \quad \text{and} \quad \mathbf{1}_{2n} = (1, \ldots, 1)^T \in \mathbb{R}^{2n}.$$

As expected, we observe from Figure 1 that the system response of the closed-loop system with the feedback matrices F and G obtained by Theorem 4.1 behaviors better than that of the original open-loop system.

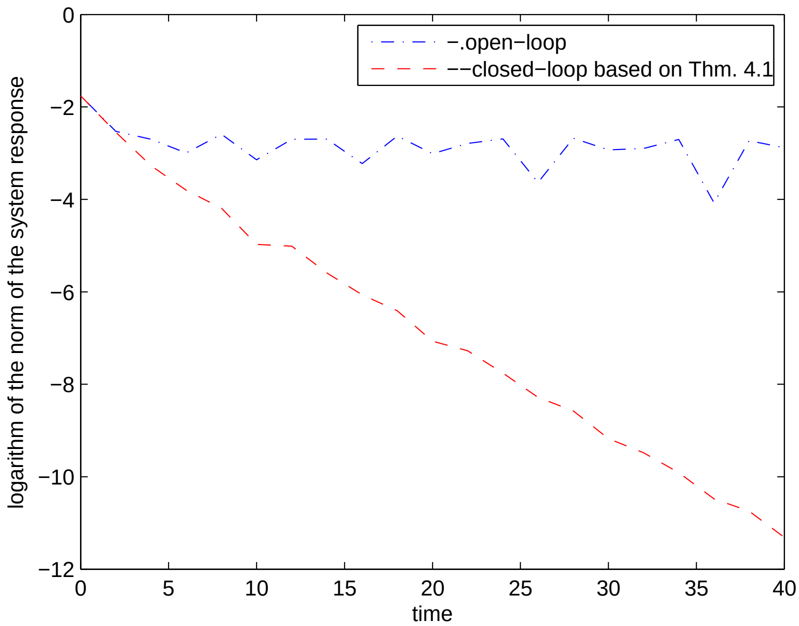
Next, we perturb the stiffness matrix K to $K + cK$ where $-0.1 \le c \le 0.1$ and the data matrices M and D, and B are kept unchanged. In this case, we compare the system responses of the open-loop system and the perturbed closed-loop systems with the feedback matrices F and G obtained by Algorithm 4.1. Figure 2 shows the base 10 logarithm of the norm of the system responses over the defined time period for different values of c. The initial condition is $\mathbf{w}(0) = 0.01 \cdot \mathbf{1}_{2n}$. We can see from Figure 2 that the system responses of the perturbed closed-loop system with the feedback matrices F and G obtained by Algorithm 4.1 are all insensitive to perturbation and successfully tend to the steady state.

5 A Hybrid Method for Partial Quadratic Eigenvalue Assignment with Time Delay

In practice, there exists time delay between the measurement of the state feedback and the implementation of feedback controller. We, therefore, would like to consider the following feedback control system with time delay τ:

$$M\ddot{\mathbf{x}}(t) + D\dot{\mathbf{x}}(t) + K\mathbf{x}(t) = B\mathbf{u}(t - \tau),$$

Figure 1: **Comparison of the system responses for Example 4.6**



where τ is the input time delay and $\mathbf{u}(t)$ is a state feedback controller defined by (3). The associated closed-loop delayed pencil is given by

$$\widetilde{P}_c(\lambda) := \lambda^2 M + \lambda(D - \mathrm{e}^{-\lambda\tau} BF^T) + (K - \mathrm{e}^{-\lambda\tau} BG^T).$$

The **Time-Delay PQEAP** is to find two feedback matrices F and G such that the closed-loop delayed pencil $\widetilde{P}_c(\lambda)$ has the desired eigenvalues $\{\mu_j\}_{j=1}^{p}$ and the $2n-p$ eigenpairs $\{(\lambda_j, \mathbf{x}_j)\}_{j=p+1}^{2n}$.

It turns out that our hybrid method for feedback norms (Theorem 4.1) and feedback norm-minimization algorithm (Algorithm 4.1) can be easily extended to the time-delay case. Without going into details, we state the time-delay versions of Theorem 4.1 and Algorithm 4.1 as follows:

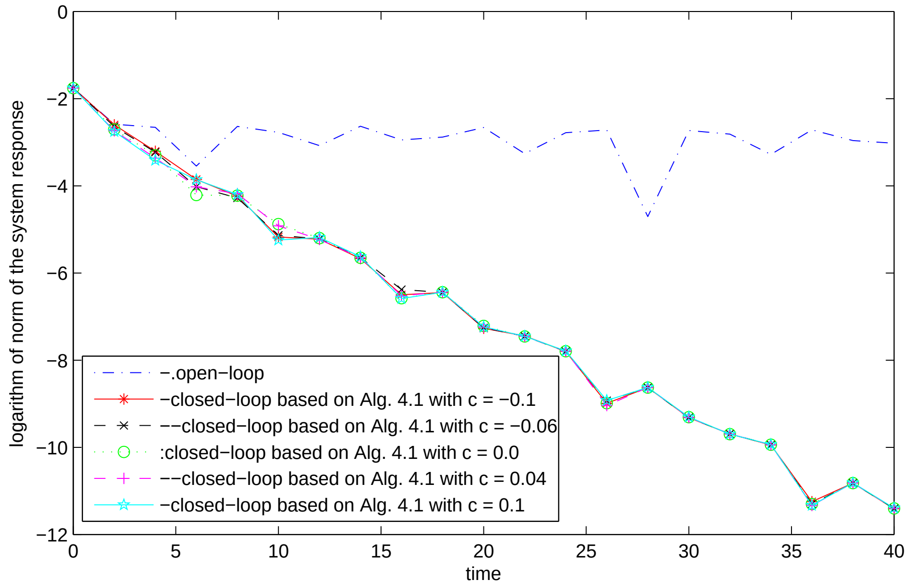
Theorem 5.1 ***Solvability of the Time-Delay PQEAP*** *Given $B \in \mathbb{R}^{n\times m}$, $\tau \geq 0$, Λ_1, X_1, and the set of p self-conjugate numbers $\{\mu_j\}_{j=1}^{p}$. Let $\Phi \in \mathbb{C}^{m\times p}$ be any matrix satisfying*

$$\det\left(\left[\mu_j B^T H(\mu_j),\ B^T H(\mu_j)\right]\begin{bmatrix} MX_1 \\ MX_1\Lambda_1 + DX_1\end{bmatrix}\Phi^T - \mathrm{e}^{\mu_j\tau} I_m\right) = 0, \quad j = 1, \ldots, p. \qquad (16)$$

Then the feedback matrices F and G defined by (4) *with the matrix Φ defined by above* (16) *solve the time-delay PQEAP.*

Proof: From Lemma 3.1 (a), we know that the feedback matrices F and G defined by (4) are such that the closed-loop time-delay pencil $\widetilde{P}_c(\lambda)$ has the $2n-p$ eigenpairs $\{(\lambda_j, \mathbf{x}_j)\}_{j=p+1}^{2n}$.

Figure 2: **Comparison of the system responses for Example 4.6 with perturbed K**



Then as in the proof of Theorem 4.1, we get

$$
\begin{aligned}
\det(\widetilde{P}_c(\mu_j)) &= \det\left(\mu_j^2 M + \mu_j(D - \mathrm{e}^{-\mu_j\tau}BF^T) + (K - \mathrm{e}^{-\mu_j\tau}BG^T)\right) \\
&= \det\left((\mu_j^2 M + \mu_j D + K) - \mathrm{e}^{-\mu_j\tau}B(G + \mu_j F)^T\right) \\
&= (\mathrm{e}^{-\mu_j\tau})^m \det(\mu_j^2 M + \mu_j D + K) \det\left(\mathrm{e}^{\mu_j\tau} I_m - (G + \mu_j F)^T H(\mu_j) B\right) \\
&= 0, \quad j = 1, \ldots, p.
\end{aligned}
$$

Therefore, the closed-loop time-delay pencil $\widetilde{P}_c(\lambda)$ contains the numbers $\{\mu_j\}_{j=1}^p$ in its spectrum.
□

Remark 5.2 *We point out that Theorem* 5.1 *generalizes the recent work by Ram, Mottershead, and Tehrani [19] in the sense that they presented a method for the single-input case with time delay only where the solution is unique whereas Theorem* 5.1 *provides a hybrid method for the multi-input case with time delay where the time-delay PQEAP has multiple solutions depending on different choice of the parameter* Φ *satisfying the conditions* (16)*. In particular, when* $m = 1$*, i.e.,* $F = \mathbf{f} \in \mathbb{R}^n$*,* $G = \mathbf{g} \in \mathbb{R}^n$*,* $B = \mathbf{b} \in \mathbb{R}^n$*, and* $\Phi = \boldsymbol{\phi}^T \in \mathbb{C}^{1\times p}$*, it is easy to check that* $\mathbf{f} = MX_1\boldsymbol{\phi}$ *and* $\mathbf{g} = (MX_1\Lambda_1 + DX_1)\boldsymbol{\phi}$*, where* $\boldsymbol{\phi}$ *is determined by*

$$
\boldsymbol{\phi} = \left(\widetilde{\Psi}\begin{bmatrix} MX_1 \\ MX_1\Lambda_1 + DX_1 \end{bmatrix}\right)^{-1} \widetilde{\mathbf{p}}
$$

with

$$\widetilde{\Psi} = \begin{bmatrix} \mu_1 \mathbf{b}^T H(\mu_1) & \mathbf{b}^T H(\mu_1) \\ \vdots & \vdots \\ \mu_p \mathbf{b}^T H(\mu_p) & \mathbf{b}^T H(\mu_p) \end{bmatrix} \quad \text{and} \quad \widetilde{\mathbf{p}} = \begin{bmatrix} e^{\mu_1 \tau} \\ \vdots \\ e^{\mu_p \tau} \end{bmatrix} \in \mathbb{C}^p.$$

The above result is similar to the one given by Ram et. al. [19] for the single-input case as in [19, Lemma 4].

Remark 5.3 *We observe from the proof of Theorem* 5.1 *that the requirements in* (16) *are equivalent to*

$$\det(\widetilde{P}_c(\mu_j)) = 0, \quad j = 1, \ldots, p.$$

We also remark that if the matrix $\Phi = [\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_p] \in \mathbb{C}^{m\times p}$ *determined by* (16) *is such that* $\boldsymbol{\phi}_j = \bar{\boldsymbol{\phi}}_k$ *if* $\lambda_j = \bar{\lambda}_k$*, then the feedback matrices* F *and* G *defined in* (4) *are real.*

From Theorem 5.1, we see that the solution to the PQEAP with time delay is not unique. The norm-minimization problem for the time-delay problem may then be stated as follows:

$$\begin{array}{ll} \min & \tilde{f}(Y) := \frac{1}{2}\,\|MX_1Y\|_F^2 + \frac{1}{2}\|(MX_1\Lambda_1 + DX_1)Y\|_F^2 \\ \text{s.t.} & \tilde{\mathbf{g}}(Y) = \mathbf{0}, \end{array} \tag{17}$$

where $\tilde{\mathbf{g}}(Y) = (\tilde{g}_1(Y), \ldots, \tilde{g}_p(Y))^T \in \mathbb{C}^p$ with $\tilde{g}_j(Y) : \mathbb{C}^{p\times m} \to \mathbb{C}$ defined by

$$\tilde{g}_j(Y) := \det\left(\left[\mu_j B^T H(\mu_j),\, B^T H(\mu_j)\right] \begin{bmatrix} MX_1 \\ MX_1\Lambda_1 + DX_1 \end{bmatrix} Y - e^{\mu_j \tau} I_m \right).$$

We note that if Y^* is the solution to Problem (17), then the minimum norm feedback matrices F and G are given by

$$F = MX_1(\Phi^*)^T \quad \text{and} \quad G = (MX_1\Lambda_1 + DX_1)(\Phi^*)^T,$$

where $\Phi^* = (Y^*)^T$.

The KKT condition for Problem (17) is to find $Y \in \mathbb{C}^{p\times m}$ and $\boldsymbol{\xi} \in \mathbb{C}^p$ such that

$$\begin{cases} \nabla \tilde{f}(Y) + \boldsymbol{\xi}^T \nabla \tilde{\mathbf{g}}(Y) = \mathbf{0}, \\ \tilde{\mathbf{g}}(Y) = \mathbf{0} \end{cases}$$

or

$$\widetilde{F}(Y, \boldsymbol{\xi}) := \begin{bmatrix} \nabla \tilde{f}(Y) + \boldsymbol{\xi}^T \nabla \tilde{\mathbf{g}}(Y) \\ \tilde{\mathbf{g}}(Y) \end{bmatrix} = \mathbf{0}. \tag{18}$$

It follows that

$$\nabla \tilde{f}(Y) = \begin{bmatrix} MX_1 \\ MX_1\Lambda_1 + DX_1 \end{bmatrix}^H \begin{bmatrix} MX_1 \\ MX_1\Lambda_1 + DX_1 \end{bmatrix} Y$$

and for $j = 1, \ldots, p$,

$$\nabla \tilde{g}_j(Y) = A_j^T \operatorname{adj}(A_j Y - e^{\mu_j \tau} I_m)^T, \quad A_j := \left[\mu_j B^T H(\mu_j),\, B^T H(\mu_j)\right] \begin{bmatrix} MX_1 \\ MX_1\Lambda_1 + DX_1 \end{bmatrix}.$$

Therefore, we may solve the nonlinear equation (18) by the classical Gauss-Newton, Levenberg-Marquardt, or trust-region-reflective methods.

Once the minimum norm solution is obtained, one may compute an eigenvector $\mathbf{z}_j$ of the closed-loop delayed pencil $\widetilde{P}_c(\lambda)$ corresponding to the new eigenvalue μ_j for $j = 1, \ldots, p$, where $(\mu_j, \mathbf{z}_j)$ satisfies

$$\left(\mu_j^2 M + \mu_j (D - \mathrm{e}^{-\mu_j \tau} B F^T) + (K - \mathrm{e}^{-\mu_j \tau} B G^T)\right) \mathbf{z}_j = \mathbf{0}. \tag{19}$$

Let $B = [\mathbf{b}_1, \ldots, \mathbf{b}_m]$, $F = [\mathbf{f}_1, \ldots, \mathbf{f}_m]$, and $G = [\mathbf{g}_1, \ldots, \mathbf{g}_m]$. For any $s \in \mathbb{C}$, define

$$\widetilde{D}_k(s) := D - \mathrm{e}^{-s\tau} \sum_{i=1}^{k} \mathbf{b}_i \mathbf{f}_i^T \quad \text{and} \quad \widetilde{K}_k(s) := K - \mathrm{e}^{-s\tau} \sum_{i=1}^{k} \mathbf{b}_i \mathbf{g}_i^T, \quad k = 1, \ldots, m$$

and $D_0 := D$ and $K_0 := K$. For any $s \in \mathbb{C}$, define the delayed receptance matrices $\widetilde{H}_k(s)$ by

$$\widetilde{H}_k(s) := \left(s^2 M + s \widetilde{D}_k(s) + \widetilde{K}_k(s)\right)^{-1}, \quad k = 0, 1, \ldots, m,$$

where $\widetilde{H}_0(s) = H(s)$. We have by the Sherman-Morrison formula,

$$\widetilde{H}_k(s) = \widetilde{H}_{k-1}(s) + \frac{\mathrm{e}^{-s\tau} \widetilde{H}_{k-1}(s) \mathbf{b}_k (\mathbf{g}_k^T + s \mathbf{f}_k^T) \widetilde{H}_{k-1}(s)}{1 - \mathrm{e}^{-s\tau} (\mathbf{g}_k^T + s \mathbf{f}_k^T) \widetilde{H}_{k-1}(s) \mathbf{b}_k}, \quad k = 1, \ldots, m. \tag{20}$$

We observe from (20) that $\widetilde{H}_k(s)$ can be computed based on $H(s)$, $\{\mathbf{f}_i\}_{i=1}^k$, $\{\mathbf{g}_i\}_{i=1}^k$, and $\mathrm{e}^{-s\tau}$.

Assume that there exists an index w such that $\mathbf{f}_w \neq \mathbf{0}$ or $\mathbf{g}_w \neq \mathbf{0}$ while $\mathbf{f}_l = \mathbf{0}$ and $\mathbf{g}_l = \mathbf{0}$ for all $l = w + 1, \ldots, m$. The relation (19) becomes

$$\left(\mu_j^2 M + \mu_j \widetilde{D}_{w-1}(\mu_j) + \widetilde{K}_{w-1}(\mu_j)\right) \mathbf{z}_j = \mathrm{e}^{-\mu_j \tau} \mathbf{b}_w (\mu_j \mathbf{f}_w^T + \mathbf{g}_w^T) \mathbf{z}_j.$$

In this case, $\tilde{\delta}_{jw} := \mathrm{e}^{-\mu_j \tau} (\mu_j \mathbf{f}_w^T + \mathbf{g}_w^T) \mathbf{z}_j$ is a nonzero scalar quantity. Therefore, one may find an eigenvector $\tilde{\mathbf{z}}_j$ corresponding to μ_j by solving

$$\left(\mu_j^2 M + \mu_j D_{w-1}(\mu_j) + K_{w-1}(\mu_j)\right) \tilde{\mathbf{z}}_j = \mathbf{b}_w, \quad \tilde{\mathbf{z}}_j = \tilde{\delta}_{jw}^{-1} \mathbf{z}_j,$$

which leads to

$$\tilde{\mathbf{z}}_j = \widetilde{H}_{w-1}(\mu_j) \mathbf{b}_w. \tag{21}$$

Based on the above discussions, we can now state the following norm-minimization algorithm for the time-delay problem.

Algorithm 5.1 (Hybrid Algorithm for Norm Minimization with Time-Delay):
Inputs:

1. The matrices $M, D \in \mathbb{R}^{n \times n}$, where $M^T = M > 0$ and $D^T = D$.

2. The control matrix $B \in \mathbb{R}^{n \times m}$ $(m \leq n)$.

3. A self-conjugate subset $\{\lambda_j\}_{j=1}^p$ of the spectrum of $P(\lambda)$ and the corresponding eigenvectors $\{\mathbf{x}_j\}_{j=1}^p$.

4. A suitably chosen self-conjugate set of numbers $\{\mu_j\}_{j=1}^p$ and the measured receptance $\{H(\mu_j)\}_{j=1}^p$.

5. τ = input time delay; ϵ = tolerance for gradient.

Outputs:

(i) The real feedback matrices F and G such that the spectrum of the close-loop delayed pencil $\widetilde{P}_c(\lambda)$ is the set $\{\mu_1, \ldots, \mu_p, \lambda_{p+1}, \ldots, \lambda_{2n}\}$ and the objective function $\tilde{f}(Y)$ defined in Problem (17) is minimized.

(ii) The closed-loop eigenvectors $\{\hat{\mathbf{z}}_j\}$ corresponding to the eigenvalues $\mu_1, \ldots, \mu_p$.

Step 1. Form the matrices Λ_1 and X_1 from the given eigenvalues and eigenvectors.

Step 2. Compute the solution Y^* to Problem (17) by solving (18). This is done by using the MATLAB function `fsolve` with the termination tolerance ϵ on the function value. This step requires $O(n^2p + m^3p^3)$ flops.

Step 3. Form the feedback matrices $F = [\mathbf{f}_1, \ldots, \mathbf{f}_m] = MX_1(\Phi^*)^T$ and $G = [\mathbf{g}_1, \ldots, \mathbf{g}_m] = (MX_1\Lambda_1 + DX_1)(\Phi^*)^T$, where $\Phi^* = (Y^*)^T$. This step needs $O(n^2p)$ operations.

Step 4. For $k = m, m-1, \ldots, 1$, determine the index w such that $\mathbf{f}_w \neq \mathbf{0}$ or $\mathbf{g}_w \neq \mathbf{0}$ while $\mathbf{f}_l = \mathbf{0}$ and $\mathbf{g}_l = \mathbf{0}$ for all $l = w+1, \ldots, m$.

Step 5. For $j = 1, \ldots, p$,

Step 5.1 Compute $\widetilde{H}_{w-1}(\mu_j)$ recursively by (20) and $\{\mathbf{f}_i\}_{i=1}^{w-1}$, $\{\mathbf{g}_i\}_{i=1}^{w-1}$, and $\mathrm{e}^{-\mu_j\tau}$. This step needs $O(n^2w)$ operations.

Step 5.2 Compute $\hat{\mathbf{z}}_j$ by (21). This step needs $O(n^2)$ operations.

We note that Algorithm 5.1 needs $O(n^2p+n^2mp+m^3p^3)$ operations totally. We also remark that the PQEAP with time delay is not considered in [9, 10, 20].

Example 5.4 ***(An Illustrative Example)*** *This example is the same as Example 4.4 with time delay* $\tau = 0.1$.

5.0.6 Numerical Experiments with the Method Based on Theorem 5.1 and Algorithm 5.1

By applying Algorithm 5.1 to Example 5.4, we have

$$\Phi^* = \begin{bmatrix} -0.0724 + 0.0286\mathrm{i} & -0.0724 - 0.0286\mathrm{i} \\ 1.0183 - 0.4028\mathrm{i} & 1.0183 + 0.4028\mathrm{i} \end{bmatrix}.$$

The corresponding minimum norm feedback matrices are given by

$$F = \begin{bmatrix} 0.1878 & -2.6421 \\ -0.4221 & 5.9368 \\ 0.3385 & -4.7610 \end{bmatrix}, \quad G = \begin{bmatrix} -1.7115 & 24.0734 \\ 3.8457 & -54.0924 \\ -3.0840 & 43.3788 \end{bmatrix}.$$

The close-loop eigenvalues and close-loop eigenvectors satisfy:

$$\begin{cases} \left|\det\left(\mu_j^2 M + \mu_j(D - \mathrm{e}^{-\mu_j\tau} BF^T) + (K - \mathrm{e}^{-\mu_j\tau} BG^T)\right)\right| < 8.0 \times 10^{-10}, & 1 \le j \le p, \\ \left\|\left(\mu_j^2 M + \mu_j(D - \mathrm{e}^{-\mu_j\tau} BF^T) + (K - \mathrm{e}^{-\mu_j\tau} BG^T)\right)\tilde{\mathbf{z}}_j\right\| < 1.7 \times 10^{-14}, & 1 \le j \le p, \\ \left\|\left(\lambda_j^2 M + \lambda_j(D - \mathrm{e}^{-\lambda_j\tau} BF^T) + (K - \mathrm{e}^{-\lambda_j\tau} BG^T)\right)\mathbf{x}_j\right\| < 1.0 \times 10^{-13}, & p+1 \le j \le 2n. \end{cases}$$

5.0.7 Comparison with Non-Optimization Method (Based on Theorem 5.1)

The non-optimization method based on Theorem 5.1 was run with two different choices of Φ. The feedback norms with one of the choices and those using the minimization algorithm, Algorithm 5.1, are displayed in Table 2.

Table 2: **Comparisons of Feedback Norms for Example 5.4 with and without Minimization**

Time-Delay Feedback Norms without Minimization		Time-Delay Feedback Norms with Minimization	
$\|F_1\|_F$	$\|F_2\|_F$	$\|F_1\|_F$	$\|F_2\|_F$
113.8812	1037	8.0760	73.5830

Thus,

- The two eigenvalues were assigned accurately.
- The eigenvalues and their corresponding eigenvectors computationally remained undamaged.
- The feedback norms using the minimization algorithm (Algorithm 5.1) were considerably smaller.

Example 5.5 *This example is the time-delay version of Example* 4.5 *with time delay* $\tau = 0.1$. *The first* $p = 2$ *eigenvalues with smallest absolute values are reassigned to* $\{-0.1, -0.2\}$ *while leaving the other eigenvalues and associated eigenvectors unchanged.*

We apply the non-optimization method based on Theorem 5.1 with one choice of Φ and Algorithm 5.1 to Example 5.5. The computed feedback norms and the errors of the closed-loop eigenvalues and eigenvectors are displayed in Table 3. Here, `tol1.`, `tol2.`, and `tol3.` stand for the upper bounds for the errors of the closed-loop eigenvalues and eigenvectors, i.e.,

$$\begin{cases} \left|\det\left(\mu_j^2 M + \mu_j(D - \mathrm{e}^{-\mu_j\tau}BF^T) + (K - \mathrm{e}^{-\mu_j\tau}BG^T)\right)\right| < \texttt{tol1.}, & 1 \le j \le p, \\ \left\|\left(\mu_j^2 M + \mu_j(D - \mathrm{e}^{-\mu_j\tau}BF^T) + (K - \mathrm{e}^{-\mu_j\tau}BG^T)\right)\tilde{\mathbf{z}}_j\right\| < \texttt{tol2.}, & 1 \le j \le p, \\ \left\|\left(\lambda_j^2 M + \lambda_j(D - \mathrm{e}^{-\lambda_j\tau}BF^T) + (K - \mathrm{e}^{-\lambda_j\tau}BG^T)\right)\mathbf{x}_j\right\| < \texttt{tol3.}, & p+1 \le j \le 2n. \end{cases}$$

Table 3: **Comparison of Time-Delay Feedback Norms with and without Minimization**

	Time-Delay Feedback Norms without Minimization		Time-Delay Feedback Norms with Minimization				
n	$\|F_1\|_F$	$\|F_2\|_F$	$\|F_1\|_F$	$\|F_2\|_F$	`tol1.`	`tol2.`	`tol3.`
10	10.8237	10.7755	1.4104	1.3891	2.3×10^{-12}	1.2×10^{-11}	2.0×10^{-14}
50	677	677	3.3969	3.3947	6.2×10^{-15}	1.5×10^{-13}	3.6×10^{-14}
100	3848	3847	4.8146	4.8138	3.0×10^{-12}	2.4×10^{-11}	1.5×10^{-12}
200	21790	21789	6.8126	6.8123	9.0×10^{-14}	4.0×10^{-13}	2.5×10^{-12}
400	123292	123291	9.6358	9.6357	8.2×10^{-12}	6.3×10^{-11}	7.4×10^{-12}

Observations: The following facts were observed based on our experiment on Example 5.4.

- The two eigenvalues were accurately reassigned and the remaining eigenvalues and eigenvectors were computationally unchanged.
- The feedback norms obtained by norm-minimization (Algorithm 5.1) were considerably smaller than those obtained by the non norm-minimization method, based on Theorem 5.1.

6 Conclusion

Active control by state feedback gives rise to partial quadratic eigenvalue assignment which concerns reassigning a few unwanted eigenvalues while keeping the remaining large number of them and the corresponding eigenvectors unchanged. For robust active control, feedback must be computed so that both feedback norms and the closed-loop eigenvalue sensitivity are minimized. We have proposed new hybrid algorithms for the partial quadratic eigenvalue assignment problem and minimization of feedback norms. Our solution methods cover systems with both with and without time delay. These hybrid methods not only make use of the system matrices but also take advantage of the receptances which are readily available from measurements. These new algorithms obviously are more efficient and offer other computational advantages over the standard methods which do not use receptances. Our future work will now be directed towards extending our hybrid method to solution of the problem of minimizing the closed-loop eigenvalue sensitivity. This is clearly a nonlinear optimization problem and is thus computationally challenging and is difficult to solve using the state-of-the-art computational techniques.

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