On Nonsingularity of Block Two-by-Two Matrices

Zheng-Jian Bai *
School of Mathematical Sciences, Xiamen University
Xiamen 361005, Fujian Province, P.R. China
Email: zjbai@xmu.edu.cn

Zhong-Zhi Bai †
State Key Laboratory of Scientific/Engineering Computing
Institute of Computational Mathematics and Scientific/Engineering Computing
Academy of Mathematics and Systems Science
Chinese Academy of Sciences, P.O. Box 2719, Beijing 100190, P.R. China
Email: bzz@lsec.cc.ac.cn

February 9, 2013

Abstract

We derive necessary and sufficient conditions for guaranteeing the nonsingularity of a block two-by-two matrix by making use of the singular value decompositions and the Moore-Penrose pseudoinverses of the matrix blocks. These conditions are complete, and much weaker and simpler than those given by Decker and Keller (J. Math. Anal. Appl. 75(1980)417-430), and may be more easily examined than those given by Bai (J. Comput. Appl. Math., 237(2013)295-306) from the computational viewpoint. We also derive general formulas for the rank of the block two-by-two matrix by utilizing either the unitarily compressed or the orthogonally projected sub-matrices.

Keywords: block two-by-two matrix, nonsingularity, rank, singular value decomposition, Moore-Penrose pseudoinverse.

AMS(MOS) Subject Classifications: 15A09, 15A18, 15A23, 65F05, 65F08, 65F10, 65F15; CR: G1.3.

*Supported by The National Natural Science Foundation (No. 11271308), The Natural Science Foundation of Fujian Province for Distinguished Young Scholars (No. 2010J06002), and The Program for New Century Excellent Talents in Universities, P.R. China.

†Supported by The Hundred Talent Project of Chinese Academy of Sciences, The National Basic Research Program (No. 2011CB309703), The National Natural Science Foundation (No. 91118001) and The National Natural Science Foundation for Creative Research Groups (No. 11021101), P.R. China.

1 Introduction

We discuss conditions that guarantee the nonsingularity of block two-by-two matrices of the form

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \tag{1.1}$$

and derive general formulas for the rank of the matrix M, where $A \in \mathbb{C}^{m\times m}$, $B \in \mathbb{C}^{m\times n}$, $C \in \mathbb{C}^{n\times m}$ and $D \in \mathbb{C}^{n\times n}$ are complex matrices. It is obvious that, under suitable partitioning, any matrix can be cast in the form (1.1).

Matrices of block two-by-two structures include as special cases the standard and the generalized saddle-point matrices [3, 4, 31, 9] and the skew-Hamiltonian matrices [23, 27]. They frequently arise from stability and bifurcation theory of ordinary differential equations [16, 17, 19, 11], order-reduction and sinc discretization of the third-order linear ordinary differential equations [26, 7], domain decomposition methods of partial differential equations [10, 28, 29, 4], finite-element discretization and first-order linearization of the two-phase flow problems based on Cahn-Hilliard equation [14, 2, 13], finite-element discretizations of PDE-constrained optimization problems [21, 25, 6], real equivalent formulations of complex linear systems [1, 6], linear and H_∞ control problems [20, 23, 24, 34, 27], matrix completions [15, 22, 32, 33], and so on.

One of the fundamental and important problems is how to examine the nonsingularity or, in general, how to determine the rank of the matrix M. When A is positive semidefinite, D is Hermitian positive semidefinite, and $C = -B^*$ have full rank, i.e., the matrix M is of the generalized saddle-point form, from [12, Theorem 3.4] we know that if

$$\mathrm{null}(\mathcal{H}(A)) \cap \mathrm{null}(B^*) = \{0\},$$

then M is nonsingular; and if M is nonsingular, then

$$\mathrm{null}(A) \cap \mathrm{null}(B^*) = \{0\}.$$

Note that the converses of the above conditions do not hold in general. so they are only either sufficient or necessary. Here the matrix A is said to be positive definite (or semidefinite) if its Hermitian part $\mathcal{H}(A) = \frac{1}{2}(A + A^*)$ is Hermitian positive definite (or semidefinite), with $(\cdot)^*$ and $\mathrm{null}(\cdot)$ denoting the conjugate transpose and the null space of the corresponding matrix, respectively; see [8]. In addition, when $D = 0$ and $m \geq n$, [12, Theorem 3.3] showed that if M is nonsingular, then

$$\mathrm{rank}(B) = n \quad \text{and} \quad \mathrm{rank}\begin{pmatrix} A \\ C \end{pmatrix} = n.$$

Note that these conditions are only necessary but sufficient.

In general, there is little result about the nonsingularity of the block two-by-two matrix M. To our knowledge, in [16] Decker and Keller proved the following result about the nonsingularity of the matrix M; see also [17].

Theorem 1.1. *[16] For the block two-by-two matrix M defined in (1.1), the following statements hold true:*

(a) *if A is singular with* $\dim(\text{null}(A)) = n \geq 1$, *then M is nonsingular if and only if*

- (a_1) $\dim(\text{range}(B)) = n$,
- (a_2) $\text{range}(A) \cap \text{range}(B) = \{0\}$,
- (a_3) $\dim(\text{range}(C)) = n$, *and*
- (a_4) $\text{null}(A) \cap \text{null}(C) = \{0\}$;

(b) *if A is nonsingular, then M is nonsingular if and only if its Schur complement $S = D - CA^{-1}B$ is nonsingular;*

(c) *if A is singular and* $\dim(\text{null}(A)) > n$, *then M is singular.*

Here range$(\cdot)$ *denotes the range space of the corresponding matrix and* dim$(\cdot)$ *the dimension of the corresponding linear space.*

Evidently, these conditions do not treat all cases as the case that A is singular with $\dim(\text{null}(A)) < n$ is excluded. Recently, in [5] Bai established the necessary and sufficient conditions for guaranteeing the nonsingularity of the matrix M, which is precisely restated in the following theorem.

Theorem 1.2. *[5] For the block two-by-two matrix M defined in (1.1), the following statements hold true:*

(a) *if A is singular, then M is nonsingular if and only if*

- (a_1) $\text{null}(A) \cap \text{null}(C) = \{0\}$,
- (a_2) $\text{null}(D) \cap \text{null}(B) = \{0\}$, *and*
- (a_3) $\text{range}\begin{pmatrix} A \\ C \end{pmatrix} \cap \text{range}\begin{pmatrix} B \\ D \end{pmatrix} = \{0\}$;

(b) *if A is nonsingular, then M is nonsingular if and only if its Schur complement $S = D - CA^{-1}B$ is nonsingular.*

In addition, for the rank of the matrix M, by utilizing the singular value decompositions and the Moore-Penrose pseudoinverses of its sub-matrices Demmel derived a range in [18] and Wei gave an expression in [30], respectively.

In this paper, we derive necessary and sufficient conditions for guaranteeing the nonsingularity of the block two-by-two matrix M by making use of either the singular value decompositions or the Moore-Penrose pseudoinverses of its sub-matrices. These conditions are complete, and much weaker and simpler than those given in [16, 17], and may be more easily examined than those given in Bai [5] from the computational viewpoint. We also derive general formulas for the rank of the block two-by-two matrix by utilizing either the unitarily compressed or the orthogonally projected sub-matrices, which further generalize and improve those shown in [18, 30].

At the end of this section, we introduce and restate some necessary notations and definitions. For a block matrix, if one of its sub-blocks is null, then the rows and columns in which they appear in that sub-block can be deleted. We use $(\cdot)^*$ to denote the conjugate transpose of either a vector or a matrix, null$(\cdot)$, range$(\cdot)$, $(\cdot)^\dagger$ and rank$(\cdot)$ to represent the null space, the range space, the Moore-Penrose pseudoinverse and the rank of the corresponding matrix, respectively, and dim$(\cdot)$ to indicate the dimension of the corresponding linear space.

2 Conditions Based on Singular Value Decompositions

In this section, by distinguishing the $(1,1)$ block $A \in \mathbb{C}^{m\times m}$ of the block two-by-two matrix $M \in \mathbb{C}^{(m+n)\times(m+n)}$ defined in (1.1) in three cases: (i) $A = 0$, (ii) $A \neq 0$ and is singular, and (iii) A is nonsingular, we establish necessary and sufficient conditions for guaranteeing the nonsingularity of M in terms of the singular value decompositions with respect to its submatrices.

Without loss of generality, throughout this section we assume that $m \leq n$ in the definition of the block two-by-two matrix $M \in \mathbb{C}^{(m+n)\times(m+n)}$ defined in (1.1) as, otherwise, we can turn to consider the equal-rank matrix

$$\widehat{M} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} M \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} D & C \\ B & A \end{bmatrix}$$

instead.

We first discuss the case that $A = 0$.

Theorem 2.1. *Assume that the* $(1,1)$ *block* $A \in \mathbb{C}^{m\times m}$ *of the block two-by-two matrix* $M \in \mathbb{C}^{(m+n)\times(m+n)}$ *defined in (1.1) is zero. Then* M *is nonsingular if and only if* B *is of full row-rank,* C *is of full column-rank, and* D_{22} *is either nonsingular or null, where*

$$\widetilde{U}^* D \widehat{V} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}, \tag{2.1}$$

and $\widehat{U}$, $\widehat{V}$ *and* $\widetilde{U}$, $\widetilde{V}$ *are unitary matrices such that*

$$\widehat{U}^* B \widehat{V} = \begin{bmatrix} B_{21} & 0 \end{bmatrix} \quad \text{and} \quad \widetilde{U}^* C \widetilde{V} = \begin{bmatrix} C_{12} \\ 0 \end{bmatrix}, \tag{2.2}$$

with $B_{21}, C_{12} \in \mathbb{C}^{m\times m}$ *being nonsingular matrix blocks, and* $D_{11} \in \mathbb{C}^{m\times m}$, $D_{22} \in \mathbb{C}^{(n-m)\times(n-m)}$, $D_{12} \in \mathbb{C}^{m\times(n-m)}$ *and* $D_{21} \in \mathbb{C}^{(n-m)\times m}$.

Proof. We first demonstrate the necessity. If the matrix M is nonsingular, then both matrices B and C must be of full ranks. Otherwise, if either B or C is rank-deficient, there must exist a nonzero vector $\widehat{x} \in \mathbb{C}^m$ or a nonzero vector $\widetilde{x} \in \mathbb{C}^m$ such that $\widehat{x}^* B = 0$ or $C\widetilde{x} = 0$, correspondingly. Let

$$\widehat{z} = \begin{bmatrix} \widehat{x} \\ 0 \end{bmatrix} \quad \text{and} \quad \widetilde{z} = \begin{bmatrix} \widetilde{x} \\ 0 \end{bmatrix}.$$

Then we have $\widehat{z} \neq 0$ and $\widetilde{z} \neq 0$. However, either $\widehat{z}^* M = 0$ or $M\widetilde{z} = 0$ holds true according to B being rank-deficient in row or C being rank-deficient in column, respectively. Therefore, M is singular, which contradicts to the nonsingularity assumption on the matrix M.

And if D_{22} is not null (i.e., $n > m$) and is singular, then there must exist a nonzero vector $\overline{y} \in \mathbb{C}^{n-m}$ such that $D_{22}\overline{y} = 0$.

Pre- and post-multiplying M by the unitary matrices

$$\begin{bmatrix} \widehat{U}^* & 0 \\ 0 & \widetilde{U}^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \widetilde{V} & 0 \\ 0 & \widehat{V} \end{bmatrix},$$

and denoting the resulting matrix as $\widehat{M}$, we know that $\widehat{M}$, being of the same rank as M, is given by

$$\widehat{M} = \begin{bmatrix} 0 & B_{21} & 0 \\ C_{12} & D_{11} & D_{12} \\ 0 & D_{21} & D_{22} \end{bmatrix}.$$

Note that B_{21} and C_{12} are nonsingular matrix blocks. Thereby, by pre- and post-multiplying $\widehat{M}$ further by the nonsingular matrices

$$\begin{bmatrix} I & 0 & 0 \\ -D_{11}B_{21}^{-1} & I & 0 \\ -D_{21}B_{21}^{-1} & 0 & I \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & 0 & -C_{12}^{-1}D_{12} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix},$$

and denoting the resulting matrix as $\overline{M}$, we know that $\overline{M}$, being of the same rank as both $\widehat{M}$ and M, is given by

$$\overline{M} = \begin{bmatrix} 0 & B_{21} & 0 \\ C_{12} & 0 & 0 \\ 0 & 0 & D_{22} \end{bmatrix}.$$

Let

$$\overline{z} = \begin{bmatrix} 0 \\ 0 \\ \overline{y} \end{bmatrix}.$$

Then we know that $\overline{z} \neq 0$ and $\overline{M}\overline{z} = 0$. Therefore, M is singular, which contradicts to the nonsingularity assumption on the matrix M, too.

Now, we demonstrate the sufficiency. If B is of full row-rank, C is of full column-rank, and D_{22} is either null (i.e., $m = n$) or nonsingular, then the matrices $\overline{M}$, $\widehat{M}$ and M defined above have the same rank, i.e., $\text{rank}(\overline{M}) = \text{rank}(\widehat{M}) = \text{rank}(M)$. Because

$$\begin{aligned} \text{rank}(M) &= \text{rank}(\overline{M}) \\ &= \text{rank}(B_{21}) + \text{rank}(C_{12}) + \text{rank}(D_{22}) \\ &= \begin{cases} m+n, & \text{if} \quad D_{22} \text{ is null (i.e., } m=n) \\ m+m+(n-m), & \text{if} \quad D_{22} \text{ is nonsingular} \end{cases} \\ &= m+n, \end{aligned}$$

we easily see that the matrix M is nonsingular. Here we have used the fact that D_{22} is null if and only if $m = n$. □

We emphasize that when $m = n$, the matrix block D_{22} is null. In this case, we have $\widetilde{U}^*D\widehat{V} = D_{11}$. It straightforwardly follows that the matrix M is nonsingular if and only if both matrices B and C are of full rank. In addition, the unitary matrices $\widehat{U}$, $\widehat{V}$ and $\widetilde{U}$, $\widetilde{V}$ can be obtained by the singular value decompositions of the matrices B and C, respectively.

Now we turn to discuss the case that $A \neq 0$ and is singular.

Theorem 2.2. *Assume the* $(1,1)$ *block* $A \in \mathbb{C}^{m\times m}$ *of the block two-by-two matrix* $M \in \mathbb{C}^{(m+n)\times(m+n)}$ *defined in (1.1) be nonzero and singular, and let* $U, V \in \mathbb{C}^{m\times m}$ *be two unitary matrices such that*

$$U^*AV = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.3}$$

with $A_{11} \in \mathbb{C}^{r\times r}$ *being nonsingular. Define*

$$U^*B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \textit{and} \quad CV = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \tag{2.4}$$

with $B_1 \in \mathbb{C}^{r\times n}$, $B_2 \in \mathbb{C}^{(m-r)\times n}$ *and* $C_1 \in \mathbb{C}^{n\times r}$, $C_2 \in \mathbb{C}^{n\times(m-r)}$. *Then* M *is nonsingular if and only if* B_2 *is of full row-rank,* C_2 *is of full column-rank, and* $D_{22} - C_{21}A_{11}^{-1}B_{12}$ *is nonsingular, where*

$$\widetilde{U}^*D\widehat{V} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}, \quad B_1\widehat{V} = \begin{bmatrix} B_{11} & B_{12} \end{bmatrix} \quad \textit{and} \quad \widetilde{U}^*C_1 = \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix}, \tag{2.5}$$

and $\widehat{U}$, $\widehat{V}$ *and* $\widetilde{U}$, $\widetilde{V}$ *are unitary matrices such that*

$$\widehat{U}^*B_2\widehat{V} = \begin{bmatrix} B_{21} & 0 \end{bmatrix} \quad \textit{and} \quad \widetilde{U}^*C_2\widetilde{V} = \begin{bmatrix} C_{12} \\ 0 \end{bmatrix}, \tag{2.6}$$

with $B_{21}, C_{12} \in \mathbb{C}^{(m-r)\times(m-r)}$ *being nonsingular, and* $D_{11} \in \mathbb{C}^{(m-r)\times(m-r)}$, $D_{22} \in \mathbb{C}^{(n+r-m)\times(n+r-m)}$, $D_{12} \in \mathbb{C}^{(m-r)\times(n+r-m)}$ *and* $D_{21} \in \mathbb{C}^{(n+r-m)\times(m-r)}$. *The other matrix blocks are* $B_{11} \in \mathbb{C}^{r\times(m-r)}$, $B_{12} \in \mathbb{C}^{r\times(n+r-m)}$, $C_{11} \in \mathbb{C}^{(m-r)\times r}$ *and* $C_{21} \in \mathbb{C}^{(n+r-m)\times r}$.

Proof. We first demonstrate the necessity. From (2.3) and (2.4) we have

$$W = \begin{bmatrix} U^* & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & B_1 \\ 0 & 0 & B_2 \\ C_1 & C_2 & D \end{bmatrix}. \tag{2.7}$$

Hence, M is nonsingular if and only if W is nonsingular.

We claim that if M is nonsingular, then both matrices B_2 and C_2 must be of full ranks. Otherwise, if either B_2 or C_2 is rank-deficient, there must exist a nonzero vector $\widehat{x} \in \mathbb{C}^{m-r}$ and a nonzero vector $\widetilde{x} \in \mathbb{C}^{m-r}$ such that $\widehat{x}^*B_2 = 0$ and $C_2\widetilde{x} = 0$, correspondingly. Let

$$\widehat{z} = \begin{bmatrix} 0 \\ \widehat{x} \\ 0 \end{bmatrix} \quad \text{and} \quad \widetilde{z} = \begin{bmatrix} 0 \\ \widetilde{x} \\ 0 \end{bmatrix}.$$

Then we know that $\widehat{z} \neq 0$ and $\widetilde{z} \neq 0$. However, either $\widehat{z}^*W = 0$ or $W\widetilde{z} = 0$ holds true according to B_2 being rank-deficient in row or C_2 being rank-deficient in column, respectively. It follows that W and, hence, M is singular, which contradicts to the nonsingularity assumption on the matrix M.

And if $D_{22} - C_{21}A_{11}^{-1}B_{12}$ is singular, then there must exist a nonzero vector $\overline{y} \in \mathbb{C}^{n+r-m}$ such that

$$(D_{22} - C_{21}A_{11}^{-1}B_{12})\overline{y} = 0.$$

By pre- and post-multiplying W by the unitary matrices

$$\begin{bmatrix} I & 0 & 0 \\ 0 & \widehat{U}^* & 0 \\ 0 & 0 & \widetilde{U}^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & 0 & 0 \\ 0 & \widetilde{V} & 0 \\ 0 & 0 & \widehat{V} \end{bmatrix},$$

and denoting the resulting matrix as $\widehat{W}$, we know that $\widehat{W}$, being of the same rank as W and M, is given by

$$\widehat{W} = \begin{bmatrix} A_{11} & 0 & B_{11} & B_{12} \\ 0 & 0 & B_{21} & 0 \\ C_{11} & C_{12} & D_{11} & D_{12} \\ C_{21} & 0 & D_{21} & D_{22} \end{bmatrix}. \tag{2.8}$$

Furthermore, by pre- and post-multiplying $\widehat{W}$ by the nonsingular matrices

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ -C_{11}A_{11}^{-1} & 0 & I & 0 \\ -C_{21}A_{11}^{-1} & 0 & 0 & I \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & 0 & -A_{11}^{-1}B_{11} & -A_{11}^{-1}B_{12} \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

and denoting the resulting matrix as $\widetilde{W}$, we know that $\widetilde{W}$, being of the same rank as M, W and $\widehat{W}$, is given by

$$\begin{aligned} \widetilde{W} &= \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ 0 & 0 & B_{21} & 0 \\ 0 & C_{12} & D_{11} - C_{11}A_{11}^{-1}B_{11} & D_{12} - C_{11}A_{11}^{-1}B_{12} \\ 0 & 0 & D_{21} - C_{21}A_{11}^{-1}B_{11} & D_{22} - C_{21}A_{11}^{-1}B_{12} \end{bmatrix} \\ &:= \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ 0 & 0 & B_{21} & 0 \\ 0 & C_{12} & X_{11} & X_{12} \\ 0 & 0 & X_{21} & D_{22} - C_{21}A_{11}^{-1}B_{12} \end{bmatrix}. \end{aligned}$$

In addition, as B_{21} and C_{12} are nonsingular, by pre- and post-multiplying $\widetilde{W}$ by the nonsingular matrices

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & -X_{11}B_{21}^{-1} & I & 0 \\ 0 & -X_{21}B_{21}^{-1} & 0 & I \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & -C_{12}^{-1}X_{12} \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

and denoting the resulting matrix as $\overline{W}$, we know that $\overline{W}$, being of the same rank as M, W, $\widehat{W}$ and $\widetilde{W}$, is given by

$$\overline{W} = \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ 0 & 0 & B_{21} & 0 \\ 0 & C_{12} & 0 & 0 \\ 0 & 0 & 0 & D_{22} - C_{21}A_{11}^{-1}B_{12} \end{bmatrix}.$$

Let

$$\overline{z} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \overline{y} \end{bmatrix}.$$

Then we know that $\overline{z} \neq 0$ and $\overline{W}\overline{z} = 0$. This shows that $\overline{W}$ and, hence, M is singular, which contradicts to the nonsingularity assumption on the matrix M, too.

Now, we demonstrate the sufficiency. If B_2 is of full row-rank, C_2 is of full column-rank, and $D_{22} - C_{21}A_{11}^{-1}B_{12}$ is nonsingular, then the matrices $\overline{W}$, $\widetilde{W}$, $\widehat{W}$, W and M defined above have the same rank, i.e.,

$$\text{rank}(\overline{W}) = \text{rank}(\widetilde{W}) = \text{rank}(\widehat{W}) = \text{rank}(W) = \text{rank}(M).$$

Because

$$\begin{aligned} \text{rank}(M) &= \text{rank}(\overline{W}) \\ &= \text{rank}(A_{11}) + \text{rank}(B_{21}) + \text{rank}(C_{12}) + \text{rank}(D_{22} - C_{21}A_{11}^{-1}B_{12}) \\ &= r + (m-r) + (m-r) + (n+r-m) \\ &= m+n, \end{aligned}$$

the matrix M is nonsingular. □

We remark that the unitary matrices U, V, $\widehat{U}$, $\widehat{V}$ and $\widetilde{U}$, $\widetilde{V}$ can be obtained by the singular value decompositions of the matrices A, B_2 and C_2, respectively.

At last, we discuss the case that A is nonsingular.

Theorem 2.3. *Assume the* $(1,1)$ *block* $A \in \mathbb{C}^{m\times m}$ *of the block two-by-two matrix* $M \in \mathbb{C}^{(m+n)\times(m+n)}$ *defined in (1.1) be nonsingular. Then* M *is nonsingular if and only if* $D - CA^{-1}B$ *is nonsingular.*

Proof. Because A is nonsingular, we have

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & -A^{-1}B \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix}.$$

Therefore, M is nonsingular if and only if $D - CA^{-1}B$ is nonsingular. □

We remark that the conclusion of Theorem 2.3 is not new and can be dated back at least to [16]; see also [17, 3, 4, 5].

3 Conditions Based on Moore-Penrose Pseudoinverses

In this section, we establish necessary and sufficient conditions for guaranteeing the nonsingularity of the block two-by-two matrix $M \in \mathbb{C}^{(m+n)\times(m+n)}$ defined in (1.1) in terms of the orthogonal projections with respect to its sub-matrices. Analogous to Section 2, without loss of generality we assume $m \leq n$ throughout this section, too.

Theorem 3.1. *The block two-by-two matrix* $M \in \mathbb{C}^{(m+n)\times(m+n)}$ *defined in (1.1) is nonsingular if and only if*

(i) $\mathrm{rank}(P) = m - \mathrm{rank}(A)$,

(ii) $\mathrm{rank}(Q) = m - \mathrm{rank}(A)$, *and*

(iii) $\mathrm{rank}(G) = n + \mathrm{rank}(A) - m$,

where

$$P = (I - AA^\dagger)B, \quad Q = C(I - A^\dagger A)$$

and

$$G = (I - QQ^\dagger)(D - CA^\dagger B)(I - P^\dagger P).$$

Proof. We demonstrate this conclusion according to three cases with respect to the $(1,1)$ block A of the matrix M: (i) $A = 0$, (ii)$A \neq 0$ and is singular, and (iii) A is nonsingular.

First, we consider the case $A = 0$. For this case, from Theorem 2.1 we know that M is nonsingular if and only if $\mathrm{rank}(B) = m$, $\mathrm{rank}(C) = m$ and $\mathrm{rank}(D_{22}) = n - m$, where D_{22} is defined in (2.1). Also, based on (2.1) and (2.2) we have $\mathrm{rank}(B_{21}) = \mathrm{rank}(C_{12}) = m$ and

$$\begin{bmatrix} 0 & B \\ C & D \end{bmatrix} = \begin{bmatrix} \widehat{U} & 0 \\ 0 & \widetilde{U} \end{bmatrix} \begin{bmatrix} 0 & B_{21} & 0 \\ C_{12} & D_{11} & D_{12} \\ 0 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \widetilde{V}^* & 0 \\ 0 & \widehat{V}^* \end{bmatrix}. \tag{3.1}$$

Partition the unitary matrices $\widetilde{U}$ and $\widehat{V}$ as

$$\widetilde{U} = \begin{bmatrix} \widetilde{U}_1 & \widetilde{U}_2 \end{bmatrix} \quad \text{and} \quad \widehat{V} = \begin{bmatrix} \widehat{V}_1 & \widehat{V}_2 \end{bmatrix},$$

with $\widetilde{U}_1, \widehat{V}_1 \in \mathbb{C}^{n\times m}$ and $\widetilde{U}_2, \widehat{V}_2 \in \mathbb{C}^{n\times(n-m)}$. Then it follows from (3.1) and $A = 0$ that

$$\begin{cases} P &= (I - AA^\dagger)B = B = \widehat{U} \begin{bmatrix} B_{21} & 0 \end{bmatrix} \widehat{V}^* = \widehat{U} B_{21} \widehat{V}_1^*, \\ Q &= C(I - A^\dagger A) = C = \widetilde{U} \begin{bmatrix} C_{12} \\ 0 \end{bmatrix} \widetilde{V}^* = \widetilde{U}_1 C_{12} \widetilde{V}^*. \end{cases} \tag{3.2}$$

By straightforward operations we have

$$I - P^\dagger P = I - \widehat{V}_1 \widehat{V}_1^* = \widehat{V}_2 \widehat{V}_2^*$$

and

$$I - QQ^\dagger = I - \widetilde{U}_1 \widetilde{U}_1^* = \widetilde{U}_2 \widetilde{U}_2^*.$$

Hence, it holds that

$$\begin{cases} B(I - P^\dagger P) &= \widehat{U} \begin{bmatrix} B_{21} & 0 \end{bmatrix} \widehat{V}^* (\widehat{V}_2 \widehat{V}_2^*) = 0, \\ (I - QQ^\dagger)C &= (\widetilde{U}_2 \widetilde{U}_2^*) \widetilde{U} \begin{bmatrix} C_{12} \\ 0 \end{bmatrix} \widetilde{V}^* = 0 \end{cases} \tag{3.3}$$

and

$$(I-QQ^\dagger)D(I-P^\dagger P)=(\widetilde{U}_2\widetilde{U}_2^*)D(\widehat{V}_2\widehat{V}_2^*)=\widetilde{U}_2D_{22}\widehat{V}_2^*. \tag{3.4}$$

The identities (3.3) and (3.4), together with (3.1), immediately lead to an explicit expression of the matrix G as follows:

$$G=(I-QQ^\dagger)(D-CA^\dagger B)(I-P^\dagger P)=\widetilde{U}_2D_{22}\widehat{V}_2^*. \tag{3.5}$$

Now the expressions of P, Q and G defined in (3.2) and (3.5) straightforwardly give the equalities

$$\operatorname{rank}(P)=\operatorname{rank}(B_{21}),\quad \operatorname{rank}(Q)=\operatorname{rank}(C_{12})\quad\text{and}\quad\operatorname{rank}(G)=\operatorname{rank}(D_{22}).$$

These equalities, together with the facts that

$$\operatorname{rank}(A)=0,\quad \operatorname{rank}(B)=m,\quad \operatorname{rank}(C)=m\quad\text{and}\quad\operatorname{rank}(D_{22})=n-m,$$

readily yield the conclusion.

Then, we consider the case $A\neq 0$ and singular. For this case, from Theorem 2.2 we know that M is nonsingular if and only if $\operatorname{rank}(B_2)=m-r$, $\operatorname{rank}(C_2)=m-r$ and $\operatorname{rank}(D_{22}-C_{21}A_{11}^{-1}B_{12})=n+r-m$, where A_{11}, B_2, C_2, B_{12}, C_{21} and D_{22} are defined in (2.3), (2.4), (2.5) and (2.6). Also, based on these unitarily compressed matrix partitions, from (2.7) and (2.8) we have

$$M=\begin{bmatrix} A & B\\ C & D\end{bmatrix}=\begin{bmatrix}\overline{U} & 0\\ 0 & \widetilde{U}\end{bmatrix}\begin{bmatrix} A_{11} & 0 & B_{11} & B_{12}\\ 0 & 0 & B_{21} & 0\\ C_{11} & C_{12} & D_{11} & D_{12}\\ C_{21} & 0 & D_{21} & D_{22}\end{bmatrix}\begin{bmatrix}\overline{V}^* & 0\\ 0 & \widehat{V}^*\end{bmatrix}, \tag{3.6}$$

where

$$\overline{U}=U\begin{bmatrix} I & 0\\ 0 & \widehat{U}\end{bmatrix}\quad\text{and}\quad \overline{V}=V\begin{bmatrix} I & 0\\ 0 & \widetilde{V}\end{bmatrix}.$$

Partition the unitary matrices $\overline{U}$, $\widetilde{U}$ and $\overline{V}$, $\widehat{V}$ as

$$\overline{U}=[\,\overline{U}_1\quad \overline{U}_2\,],\quad \widetilde{U}=\begin{bmatrix}\widetilde{U}_1 & \widetilde{U}_2\end{bmatrix}$$

and

$$\overline{V}=[\,\overline{V}_1\quad \overline{V}_2\,],\quad \widehat{V}=\begin{bmatrix}\widehat{V}_1 & \widehat{V}_2\end{bmatrix},$$

with $\overline{U}_1,\overline{V}_1\in\mathbb{C}^{m\times r}$, $\overline{U}_2,\overline{V}_2\in\mathbb{C}^{m\times(m-r)}$, $\widetilde{U}_1,\widehat{V}_1\in\mathbb{C}^{n\times(m-r)}$ and $\widetilde{U}_2,\widehat{V}_2\in\mathbb{C}^{n\times(n+r-m)}$. Then it follows from (3.6) that

$$P=(I-AA^\dagger)B=\overline{U}_2B_{21}\widehat{V}_1^*\quad\text{and}\quad Q=C(I-A^\dagger A)=\widetilde{U}_1C_{12}\overline{V}_2^*, \tag{3.7}$$

where we have used the expressions

$$A=\overline{U}_1A_{11}\overline{V}_1^*,\quad AA^\dagger=\overline{U}_1\overline{U}_1^*\quad\text{and}\quad A^\dagger A=\overline{V}_1\overline{V}_1^*.$$

By straightforward operations we have

$$I - P^\dagger P = I - \widehat{V}_1\widehat{V}_1^* = \widehat{V}_2\widehat{V}_2^*$$

and

$$I - QQ^\dagger = I - \widetilde{U}_1\widetilde{U}_1^* = \widetilde{U}_2\widetilde{U}_2^*.$$

Hence, it holds that

$$\begin{cases} B(I - P^\dagger P) &= U\begin{bmatrix} I & 0 \\ 0 & \widehat{U}\end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & 0\end{bmatrix}\widehat{V}^*(\widehat{V}_2\widehat{V}_2^*) = \overline{U}_1 B_{12}\widehat{V}_2^*, \\ (I - QQ^\dagger)C &= (\widetilde{U}_2\widetilde{U}_2^*)\widetilde{U}\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & 0\end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & \widetilde{V}^*\end{bmatrix}V^* = \widetilde{U}_2 C_{21}\overline{V}_1^* \end{cases} \tag{3.8}$$

and

$$(I - QQ^\dagger)D(I - P^\dagger P) = \widetilde{U}_2(\widetilde{U}_2^* D\widehat{V}_2)\widehat{V}_2^* = \widetilde{U}_2 D_{22}\widehat{V}_2^*. \tag{3.9}$$

The identities (3.8) and (3.9), together with (3.6), immediately lead to an explicit expression of the matrix G as follows:

$$G = (I - QQ^\dagger)(D - CA^\dagger B)(I - P^\dagger P) = \widetilde{U}_2(D_{22} - C_{21}A_{11}^{-1}B_{12})\widehat{V}_2^*. \tag{3.10}$$

Now the expressions of P, Q and G defined in (3.7) and (3.10) straightforwardly give the equalities

$$\text{rank}(P) = \text{rank}(B_{21}), \quad \text{rank}(Q) = \text{rank}(C_{12}) \quad \text{and} \quad \text{rank}(G) = \text{rank}(D_{22} - C_{21}A_{11}^{-1}B_{12}).$$

These equalities, together with the facts that

$$\text{rank}(A) = \text{rank}(A_{11}) = r, \quad \text{rank}(B_2) = m - r, \quad \text{rank}(C_2) = m - r$$

and

$$\text{rank}(D_{22} - C21A_{11}^{-1}B_{12}) = n + r - m,$$

readily yield the conclusion.

At last, we consider the case that A is nonsingular. For this case, from Theorem 2.3 we know that M is nonsingular if and only if $D - CA^{-1}B$ is nonsingular. As now $A^\dagger = A^{-1}$, by straightforward operations we have

$$P = (I - AA^\dagger)B = 0, \quad Q = C(I - A^\dagger A) = 0$$

and

$$G = (I - QQ^\dagger)(D - CA^\dagger B)(I - P^\dagger P) = D - CA^{-1}B.$$

Hence, it holds that

$$\text{rank}(A) = m, \quad \text{rank}(P) = 0, \quad \text{rank}(Q) = 0$$

and

$$\text{rank}(G) = \text{rank}(D - CA^{-1}B) = n,$$

which shows the validity of the conclusion. □

4 General Formulas for Matrix Ranks

In this section, we establish general formulas for the rank of the block two-by-two matrix $M \in \mathbb{C}^{(m+n)\times(m+n)}$ defined in (1.1) in terms of the singular value decompositions and the orthogonal projections with respect to its sub-matrices.

Theorem 4.1. *For the block two-by-two matrix $M \in \mathbb{C}^{(m+n)\times(m+n)}$ defined in (1.1), let $U, V \in \mathbb{C}^{m\times m}$ be two unitary matrices such that*

$$U^*AV = \begin{bmatrix} A_{11} & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.1}$$

with A_{11} being either null or a nonsingular $r \times r$ matrix block, and define

$$U^*B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \text{and} \quad CV = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \tag{4.2}$$

with B_1 being either null or a $r \times n$ matrix block, C_1 being either null or an $n \times r$ matrix block, correspondingly, and $B_2 \in \mathbb{C}^{(m-r)\times n}$ and $C_2 \in \mathbb{C}^{n\times(m-r)}$. In addition, let $\widehat{U} \in \mathbb{C}^{(m-r)\times(m-r)}$, $\widehat{V} \in \mathbb{C}^{n\times n}$ and $\widetilde{U} \in \mathbb{C}^{n\times n}$, $\widetilde{V} \in \mathbb{C}^{(m-r)\times(m-r)}$ be unitary matrices such that

$$\widehat{U}^*B_2\widehat{V} = \begin{bmatrix} B_{21} & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \widetilde{U}^*C_2\widetilde{V} = \begin{bmatrix} C_{12} & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.3}$$

with $B_{21} \in \mathbb{C}^{s_1\times s_1}$, $C_{12} \in \mathbb{C}^{s_2\times s_2}$ being either null (i.e., $s_1 = 0$, $s_2 = 0$) or nonsingular, and define

$$B_1\widehat{V} = \begin{bmatrix} B_{11} & B_{12} \end{bmatrix}, \quad \widetilde{U}^*C_1 = \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} \quad \text{and} \quad \widetilde{U}^*D\widehat{V} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}, \tag{4.4}$$

with $B_{11} \in \mathbb{C}^{r\times s_1}$, $B_{12} \in \mathbb{C}^{r\times(n-s_1)}$, $C_{11} \in \mathbb{C}^{s_2\times r}$, $C_{21} \in \mathbb{C}^{(n-s_2)\times r}$, and $D_{11} \in \mathbb{C}^{s_2\times s_1}$, $D_{22} \in \mathbb{C}^{(n-s_2)\times(n-s_1)}$, $D_{12} \in \mathbb{C}^{s_2\times(n-s_1)}$ and $D_{21} \in \mathbb{C}^{(n-s_2)\times s_1}$. Then it holds that

$$\operatorname{rank}(M) = \operatorname{rank}(A_{11}) + \operatorname{rank}(B_{21}) + \operatorname{rank}(C_{12}) + \operatorname{rank}(D_{22} - C_{21}A_{11}^{-1}B_{12}). \tag{4.5}$$

Here we should replace $D_{22} - C_{21}A_{11}^{-1}B_{12}$ by D_{22} if A_{11} is null, and set $\operatorname{rank}(A_{11})$, $\operatorname{rank}(B_{21})$ *or* $\operatorname{rank}(C_{12})$ *to be zero if the corresponding matrix block A_{11}, B_{21} or C_{12} is null.*

Proof. From (4.1) and (4.2) we have

$$\begin{bmatrix} U^* & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & B_1 \\ 0 & 0 & B_2 \\ C_1 & C_2 & D \end{bmatrix},$$

and using (4.3) and (4.4), after direct operations we obtain

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \overline{U} & 0 \\ 0 & \widetilde{U} \end{bmatrix}\begin{bmatrix} A_{11} & 0 & 0 & B_{11} & B_{12} \\ 0 & 0 & 0 & B_{21} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ C_{11} & C_{12} & 0 & D_{11} & D_{12} \\ C_{21} & 0 & 0 & D_{21} & D_{22} \end{bmatrix}\begin{bmatrix} \overline{V}^* & 0 \\ 0 & \widehat{V}^* \end{bmatrix}, \tag{4.6}$$

where

$$\overline{U} = U \begin{bmatrix} I & 0 \\ 0 & \widehat{U} \end{bmatrix} \quad \text{and} \quad \overline{V} = V \begin{bmatrix} I & 0 \\ 0 & \widetilde{V} \end{bmatrix}.$$

Denote by

$$\widehat{W} = \begin{bmatrix} A_{11} & 0 & 0 & B_{11} & B_{12} \\ 0 & 0 & 0 & B_{21} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ C_{11} & C_{12} & 0 & D_{11} & D_{12} \\ C_{21} & 0 & 0 & D_{21} & D_{22} \end{bmatrix}.$$

Then we see that the matrices M and $\widehat{W}$ have the same rank.

When A_{11} is null, according to (4.6) we know that M has the same rank as the matrix

$$\begin{bmatrix} 0 & B_{21} & 0 \\ C_{12} & D_{11} & D_{12} \\ 0 & D_{21} & D_{22} \end{bmatrix},$$

which can be reduced to

$$\begin{bmatrix} C_{12} & D_{12} \\ 0 & D_{22} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} B_{21} & 0 \\ D_{21} & D_{22} \end{bmatrix}$$

if either B_{21} or C_{12} is null, and can be transformed to

$$\begin{bmatrix} 0 & B_{21} & 0 \\ C_{12} & 0 & 0 \\ 0 & 0 & D_{22} \end{bmatrix}$$

by pre- and post-multiplying nonsingular matrices

$$\begin{bmatrix} I & 0 & 0 \\ -D_{11}B_{21}^{-1} & I & 0 \\ -D_{21}B_{21}^{-1} & 0 & I \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & 0 & C_{12}^{-1}D_{12} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

if both B_{21} and C_{12} are nonsingular. Hence,

$$\mathrm{rank}(M) = \mathrm{rank}(B_{21}) + \mathrm{rank}(C_{12}) + \mathrm{rank}(D_{22}),$$

which shows that the rank formula (4.5) is true.

When A_{11} is nonsingular, by pre- and post-multiplying $\widehat{W}$ by the nonsingular matrices

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ -C_{11}A_{11}^{-1} & 0 & 0 & I & 0 \\ -C_{21}A_{11}^{-1} & 0 & 0 & 0 & I \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & 0 & 0 & -A_{11}^{-1}B_{11} & -A_{11}^{-1}B_{12} \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

and denoting the resulting matrix as $\widetilde{W}$, we know that $\widetilde{W}$, having the same rank as M and $\widehat{W}$, is given by

$$\widetilde{W} = \begin{bmatrix} A_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{21} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & C_{12} & 0 & D_{11} - C_{11}A_{11}^{-1}B_{11} & D_{12} - C_{11}A_{11}^{-1}B_{12} \\ 0 & 0 & 0 & D_{21} - C_{21}A_{11}^{-1}B_{11} & D_{22} - C_{21}A_{11}^{-1}B_{12} \end{bmatrix}$$
$$:= \begin{bmatrix} A_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{21} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & C_{12} & 0 & X_{11} & X_{12} \\ 0 & 0 & 0 & X_{21} & D_{22} - C_{21}A_{11}^{-1}B_{12} \end{bmatrix}.$$

Evidently, if either B_{21} or C_{12} is null, the matrix M has the same rank as the matrices

$$\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & C_{12} & X_{12} \\ 0 & 0 & D_{22} - C_{21}A_{11}^{-1}B_{12} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & B_{21} & 0 \\ 0 & X_{21} & D_{22} - C_{21}A_{11}^{-1}B_{12} \end{bmatrix}.$$

Hence, the rank formula (4.5) holds true. And if both B_{21} and C_{12} are nonsingular, by pre- and post-multiplying $\widetilde{W}$ further by the nonsingular matrices

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & -X_{11}B_{21}^{-1} & 0 & I & 0 \\ 0 & -X_{21}B_{21}^{-1} & 0 & 0 & I \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & -C_{12}^{-1}X_{12} \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

and denoting the resulting matrix as $\overline{W}$, we know that $\overline{W}$, having the same rank as M, $\widehat{W}$ and $\widetilde{W}$, is given by

$$\overline{W} = \begin{bmatrix} A_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{21} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & C_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{22} - C_{21}A_{11}^{-1}B_{12} \end{bmatrix}.$$

Therefore, the rank formula (4.5) is valid, too. □

Theorem 4.2. *For the block two-by-two matrix $M \in \mathbb{C}^{(m+n)\times(m+n)}$ defined in (1.1), it holds that*

$$\text{rank}(M) = \text{rank}(A) + \text{rank}(P) + \text{rank}(Q) + \text{rank}(G), \tag{4.7}$$

where

$$P = (I - AA^\dagger)B, \quad Q = C(I - A^\dagger A)$$

and

$$G = (I - QQ^\dagger)(D - CA^\dagger B)(I - P^\dagger P).$$

Proof. We adopt the same matrix decompositions (4.1)-(4.4) as in Theorem 4.1, and denote by

$$\widetilde{U} = \left[\begin{array}{cc} \widetilde{U}_1 & \widetilde{U}_2 \end{array} \right] \quad \text{and} \quad \widehat{V} = \left[\begin{array}{cc} \widehat{V}_1 & \widehat{V}_2 \end{array} \right],$$

with

$$\widetilde{U}_1 \in \mathbb{C}^{n\times s_2}, \quad \widetilde{U}_2 \in \mathbb{C}^{n\times(n-s_2)}, \quad \widehat{V}_1 \in \mathbb{C}^{n\times s_1} \quad \text{and} \quad \widehat{V}_2 \in \mathbb{C}^{n\times(n-s_1)}.$$

Then it follows from (4.5) that to prove (4.7) we only need to demonstrate the identities

$$\operatorname{rank}(A) = \operatorname{rank}(A_{11}), \quad \operatorname{rank}(P) = \operatorname{rank}(B_{21}), \quad \operatorname{rank}(Q) = \operatorname{rank}(C_{12}) \tag{4.8}$$

and

$$\operatorname{rank}(G) = \operatorname{rank}(D_{22} - C_{21}A_{11}^{-1}B_{12}). \tag{4.9}$$

From (4.1) we immediately have $\operatorname{rank}(A) = \operatorname{rank}(A_{11})$, so the first identity in (4.8) is valid. In addition, it holds that

$$A = U \left[\begin{array}{cc} A_{11} & 0 \\ 0 & 0 \end{array} \right] V^* \quad \text{and} \quad A^\dagger = V \left[\begin{array}{cc} A_{11}^{-1} & 0 \\ 0 & 0 \end{array} \right] U^*. \tag{4.10}$$

If A_{11} is null, then $A^\dagger = 0$, $I - AA^\dagger = I - A^\dagger A = I$, and

$$P = B = B_2, \quad Q = C = C_2 \quad \text{and} \quad G = (I - C_2C_2^\dagger)D(I - B_2^\dagger B_2).$$

Whether B_{21} or C_{12} is null or not, in accordance with (4.3) we can obtain

$$\operatorname{rank}(P) = \operatorname{rank}(B_2) = \operatorname{rank}(B_{21}) \quad \text{and} \quad \operatorname{rank}(Q) = \operatorname{rank}(C_2) = \operatorname{rank}(C_{12}),$$

which give the last two identities in (4.8). So we only need to verify the validity of the identity (4.9), which, in this case, is reduced to

$$\operatorname{rank}(G) = \operatorname{rank}(D_{22}).$$

In fact, if both B_{21} and C_{12} are null, then $B_2 = 0$, $C_2 = 0$ and $\widetilde{U}^*D\widehat{V} = D_{22}$. Hence, $G = D = \widetilde{U}D_{22}\widehat{V}^*$, which implies $\operatorname{rank}(G) = \operatorname{rank}(D_{22})$.

If B_{21} is null and C_{12} is nonsingular, then $B_2 = 0$,

$$\widetilde{U}^*D\widehat{V} = \left[\begin{array}{c} D_{12} \\ D_{22} \end{array} \right], \quad C_2 = \widetilde{U} \left[\begin{array}{cc} C_{12} & 0 \\ 0 & 0 \end{array} \right] \widehat{V}^*, \quad C_2^\dagger = \widehat{V} \left[\begin{array}{cc} C_{12}^{-1} & 0 \\ 0 & 0 \end{array} \right] \widetilde{U}^*$$

and

$$\begin{aligned} G &= (I - C_2C_2^\dagger)D = \left(I - \widetilde{U} \left[\begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right] \widetilde{U}^* \right) D = \widetilde{U} \left[\begin{array}{cc} 0 & 0 \\ 0 & I \end{array} \right] \widetilde{U}^*D \\ &= \widetilde{U} \left[\begin{array}{cc} 0 & 0 \\ 0 & I \end{array} \right] (\widetilde{U}^*D\widehat{V})\widehat{V}^* = \widetilde{U} \left[\begin{array}{cc} 0 & 0 \\ 0 & I \end{array} \right] \left[\begin{array}{c} D_{12} \\ D_{22} \end{array} \right] \widehat{V}^* = \widetilde{U} \left[\begin{array}{c} 0 \\ D_{22} \end{array} \right] \widehat{V}^*, \end{aligned}$$

which leads to $\text{rank}(G) = \text{rank}(D_{22})$.

If C_{12} is null and B_{21} is nonsingular, then $C_2 = 0$,

$$\widetilde{U}^* D \widehat{V} = \begin{bmatrix} D_{21} & D_{22} \end{bmatrix}, \quad B_2 = \widehat{U} \begin{bmatrix} B_{21} & 0 \\ 0 & 0 \end{bmatrix} \widehat{V}^*, \quad B_2^\dagger = \widehat{V} \begin{bmatrix} B_{21}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \widehat{U}^*$$

and, analogously,

$$\begin{aligned} G &= D(I - B_2^\dagger B_2) = D\left(I - \widehat{V} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \widehat{V}^*\right) = D\widehat{V} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \widehat{V}^* \\ &= \widetilde{U}(\widetilde{U}^* D \widehat{V}) \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \widehat{V}^* = \widetilde{U} \begin{bmatrix} D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \widehat{V}^* = \widetilde{U} \begin{bmatrix} 0 & D_{22} \end{bmatrix} \widehat{V}^*, \end{aligned}$$

which results in $\text{rank}(G) = \text{rank}(D_{22})$.

And if both B_{21} and C_{12} are nonsingular, then

$$B_2 = \widehat{U} \begin{bmatrix} B_{21} & 0 \\ 0 & 0 \end{bmatrix} \widehat{V}^*, \quad B_2^\dagger = \widehat{V} \begin{bmatrix} B_{21}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \widehat{U}^*, \quad B_2^\dagger B_2 = \widehat{V} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \widehat{V}^*$$

and

$$C_2 = \widetilde{U} \begin{bmatrix} C_{12} & 0 \\ 0 & 0 \end{bmatrix} \widetilde{V}^*, \quad C_2^\dagger = \widetilde{V} \begin{bmatrix} C_{12}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \widetilde{U}^*, \qquad C_2 C_2^\dagger = \widetilde{U} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \widetilde{U}^*.$$

By direct operations we have

$$G = (I - C_2 C_2^\dagger) D (I - B_2^\dagger B_2) = \widetilde{U} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \widetilde{U}^* D \widehat{V} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \widehat{V}^* = \widetilde{U}_2 D_{22} \widehat{V}_2^*,$$

which gives $\text{rank}(G) = \text{rank}(D_{22})$.

If A_{11} is nonsingular, from (4.10) we can get

$$I - AA^\dagger = U \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} U^* \quad \text{and} \quad I - A^\dagger A = V \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} V^*.$$

In accordance with (4.2) and (4.3) we obtain

$$\begin{aligned} P &= U \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} U^* B = U \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = U \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \\ &= U \begin{bmatrix} I & 0 \\ 0 & \widehat{U} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ B_{21} & 0 \\ 0 & 0 \end{bmatrix} \widehat{V}^* \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} Q &= CV \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} V^* = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} V^* = \begin{bmatrix} 0 & C_2 \end{bmatrix} V^* \\ &= \widetilde{U} \begin{bmatrix} 0 & C_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \widetilde{V}^* \end{bmatrix} V^*, \end{aligned} \tag{4.12}$$

which give the last two identities in (4.8). So we only need to verify the validity of the identity (4.9).

As a matter of fact, if both B_{21} and C_{12} are nonsingular, it follows from (4.11) and (4.12) that

$$I - P^\dagger P = \widehat{V} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \widehat{V}^* = \widehat{V}_2 \widehat{V}_2^*$$

and

$$I - QQ^\dagger = \widetilde{U} \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \widetilde{U}^* = \widetilde{U}_2 \widetilde{U}_2^*.$$

Thereby, by making use of (4.1) and (4.4) we obtain

$$\begin{aligned} G &= (\widetilde{U}_2 \widetilde{U}_2^*)(D - CA^\dagger B)(\widehat{V}_2 \widehat{V}_2^*) \\ &= (\widetilde{U}_2 \widetilde{U}_2^*)\widetilde{U}[(\widetilde{U}^* D \widehat{V}) - (\widetilde{U}^* CV)(U^* AV)^\dagger (U^* B \widehat{V})]\widehat{V}^*(\widehat{V}_2 \widehat{V}_2^*) \\ &= \begin{bmatrix} 0 & \widetilde{U}_2 \end{bmatrix} (\widetilde{U}^* D \widehat{V} - \widetilde{U}^* C_1 A_{11}^{-1} B_1 \widehat{V}) \begin{bmatrix} 0 \\ \widehat{V}_2^* \end{bmatrix} \\ &= \widetilde{U}_2 (D_{22} - C_{21} A_{11}^{-1} B_{12}) \widehat{V}_2^*, \end{aligned} \tag{4.13}$$

which implies $\text{rank}(G) = \text{rank}(D_{22} - C_{21} A_{11}^{-1} B_{12})$.

If both B_{21} and C_{12} are null, then $B_2 = 0$, $C_2 = 0$, $B_1 \widehat{V} = B_{12}$, $\widetilde{U}^* C_1 = C_{21}$. Hence, $P = 0$, $Q = 0$, $\widetilde{U}^* D \widehat{V} = D_{22}$, and

$$\begin{aligned} G &= D - CA^\dagger B = D - CV(U^* AV)^\dagger U^* B \\ &= D - \begin{bmatrix} C_1 & 0 \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = D - C_1 A_{11}^{-1} B_1 \\ &= \widetilde{U}(\widetilde{U}^* D \widehat{V} - \widetilde{U}^* C_1 A_{11}^{-1} B_1 \widehat{V})\widehat{V}^* = \widetilde{U}(D_{22} - C_{21} A_{11}^{-1} B_{12})\widehat{V}^*, \end{aligned}$$

which leads to $\text{rank}(G) = \text{rank}(D_{22} - C_{21} A_{11}^{-1} B_{12})$.

If B_{21} is nonsingular and C_{12} is null, then $C_2 = 0$ and $\widetilde{U}^* C_1 = C_{21}$. Hence,

$$Q = 0, \quad D - CA^\dagger B = D - C_1 A_{11}^{-1} B_1, \quad \widetilde{U}^* D \widehat{V} = \begin{bmatrix} D_{21} & D_{22} \end{bmatrix}$$

and, similarly to (4.11), it holds that

$$P = U \begin{bmatrix} I & 0 \\ 0 & \widehat{U} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ B_{21} & 0 \\ 0 & 0 \end{bmatrix} \widehat{V}^*.$$

As now $I - P^\dagger P = \widehat{V}_2 \widehat{V}_2^*$, analogously to (4.13) we straightforwardly have

$$\begin{aligned} G &= (D - C_1 A_{11}^{-1} B_1)(\widehat{V}_2 \widehat{V}_2^*) \\ &= \widetilde{U}(\widetilde{U}^* D \widehat{V} - \widetilde{U}^* C_1 A_{11}^{-1} B_1 \widehat{V})\widehat{V}^*(\widehat{V}_2 \widehat{V}_2^*) \\ &= \widetilde{U} \begin{bmatrix} D_{21} - C_{21} A_{11}^{-1} B_{11} & D_{22} - C_{21} A_{11}^{-1} B_{12} \end{bmatrix} \begin{bmatrix} 0 \\ \widehat{V}_2^* \end{bmatrix} \\ &= \widetilde{U}(D_{22} - C_{21} A_{11}^{-1} B_{12})\widehat{V}_2^*, \end{aligned}$$

which results in $\mathrm{rank}(G) = \mathrm{rank}(D_{22} - C_{21}A_{11}^{-1}B_{12})$.

And if B_{21} is null and C_{12} is nonsingular, then $B_2 = 0$ and $B_1\widehat{V} = B_{12}$. Hence,

$$P = 0, \quad D - CA^{\dagger}B = D - C_1A_{11}^{-1}B_1, \quad \widetilde{U}^*D\widehat{V} = \begin{bmatrix} D_{12} \\ D_{22} \end{bmatrix},$$

and, similarly to (4.12), it holds that

$$Q = \widetilde{U} \begin{bmatrix} 0 & C_{12} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \widetilde{V}^* \end{bmatrix} V^*.$$

As now $I - QQ^{\dagger} = \widetilde{U}_2\widetilde{U}_2^*$, analogously to (4.13) we immediately have

$$\begin{aligned} G &= (\widetilde{U}_2\widetilde{U}_2^*)(D - C_1A_{11}^{-1}B_1) \\ &= (\widetilde{U}_2\widetilde{U}_2^*)\widetilde{U}(\widetilde{U}^*D\widehat{V} - \widetilde{U}^*C_1A_{11}^{-1}B_1\widehat{V})\widehat{V}^* \\ &= \begin{bmatrix} 0 & \widetilde{U}_2 \end{bmatrix} \begin{bmatrix} D_{12} - C_{11}A_{11}^{-1}B_{12} \\ D_{22} - C_{21}A_{11}^{-1}B_{12} \end{bmatrix} \widehat{V}^* \\ &= \widetilde{U}_2(D_{22} - C_{21}A_{11}^{-1}B_{12})\widehat{V}^*, \end{aligned}$$

which gives $\mathrm{rank}(G) = \mathrm{rank}(D_{22} - C_{21}A_{11}^{-1}B_{12})$. □

Theorem 4.2 is a special case of Theorem 2.1 in [30], which straightforwardly yields the following result about the nonsingularity of the block two-by-two matrix $M \in \mathbb{C}^{(m+n)\times(m+n)}$ defined in (1.1).

Theorem 4.3. *The block two-by-two matrix $M \in \mathbb{C}^{(m+n)\times(m+n)}$ defined in (1.1) is nonsingular if and only if*

$$\mathrm{rank}(A) + \mathrm{rank}(P) + \mathrm{rank}(Q) + \mathrm{rank}(G) = m + n.$$

5 Concluding Remarks

In the most general fashion, we have established necessary and sufficiency conditions for the nonsingularity of the block two-by-two matrix. The basic tool used in our approach is the singular value decomposition of matrix, which is computationally feasible and numerically stable. The new results obtained in this paper include the existing ones such as those given in [12] as special cases, and could be equivalent to those presented in [5]. We have also derived formulas for the rank of the block two-by-two matrix in terms of either the unitary compressions (via the singular value decompositions) or the orthogonal projections (via the Moore-Penrose pseudoinverses) of the involved sub-matrices, which generalize the rank formula and, in particular, result in the alternative nonsingularity conditions given in [30]. However, sharply estimating the condition number and the eigenvalue bounds for a block two-by-two matrix under reasonably weak restrictions are still open problems, which are theoretically important and practically useful; see [3, 4, 7].

References

[1] O. Axelsson and A. Kucherov, Real valued iterative methods for solving complex symmetric linear systems, *Numer. Linear Algebra Appl.*, 7(2000), 197-218.

[2] V.E. Badalassi, H.D. Ceniceros and S. Banerjee, Computation of multiphase systems with phase field models, *J. Comput. Phys.*, 190(2003), 371-397.

[3] Z.-Z. Bai, Construction and analysis of structured preconditioners for block two-by-two matrices, *J. Shanghai Univ. (English Edition)*, 8(2004), 397-405.

[4] Z.-Z. Bai, Structured preconditioners for nonsingular matrices of block two-by-two structures, *Math. Comput.*, 75(2006), 791-815.

[5] Z.-Z. Bai, Eigenvalue estimates for saddle point matrices of Hermitian and indefinite leading blocks, *J. Comput. Appl. Math.*, 237(2013), 295-306.

[6] Z.-Z. Bai, M. Benzi, F. Chen and Z.-Q. Wang, Preconditioned MHSS iteration methods for a class of block two-by-two linear systems with applications to distributed control problems, *IMA J. Numer. Anal.*, 33(2013), 343-369.

[7] Z.-Z. Bai, R.H. Chan and Z.-R. Ren, On order-reducible sinc discretizations and block-diagonal preconditioning methods for linear third-order ordinary differential equations, *Numer. Linear Algebra Appl.*, In Print, (2013).

[8] Z.-Z. Bai, G.H. Golub and M.K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, *SIAM J. Matrix Anal. Appl.*, 24(2003), 603-626.

[9] Z.-Z. Bai, M.K. Ng and Z.-Q. Wang, Constraint preconditioners for symmetric indefinite matrices, *SIAM J. Matrix Anal. Appl.*, 31(2009), 410-433.

[10] Z.-Z. Bai, J.-C. Sun and D.-R. Wang, A unified framework for the construction of various matrix multisplitting iterative methods for large sparse system of linear equations, *Computers Math, Appl.*, 32(1996), 51-76.

[11] L. Barreira and C. Valls, Ordinary Differential Equations: Qualitative Theory, *American Mathematical Society*, Providence, Rhode Island, 2012.

[12] M. Benzi, G.H. Golub and J. Liesen, Numerical solution of saddle point problems *Acta Numer.*, 14(2005), 1-137.

[13] A. Berti and I. Bochicchio, A mathematical model for phase separation: A generalized Cahn-Hilliard equation, *Math. Meth. Appl. Sci.*, 34(2011), 1193-1201.

[14] J.W. Cahn and J.E. Hilliard, Free energy of a nonuniform system. I. Interfacial free energy, *J. Chem. Phys.*, 28(1958), 258-267.

[15] H. Dai, Completing a symmetric 2×2 block matrix and its inverse, *Linear Algebra Appl.*, 235(1996), 235-245.

[16] D.W. Decker and H.B. Keller, Multiple limit point bifurcation, *J. Math. Anal. Appl.*, 75(1980), 417-430.

[17] D.W. Decker and H.B. Keller, Path following near bifurcation, *Commun. Pure Appl. Math.*, 34(1981), 149-175.

[18] J.W. Demmel, The smallest perturbation of a submatrix which lowers the rank and constrained total least squares problems, *SIAM J. Numer. Anal.*, 24(1987), 199-206.

[19] G. Iooss and D.D. Joseph, Elementary Stability and Bifurcation Theory, *Springer-Verlag*, Berlin and Heidelberg, 1990.

[20] T. Kailath, Linear Systems, *Prentice-Hall*, Englewood Cliffs, New Jersey, 1980.

[21] J.L. Lions, Optimal Control of Systems Governed by Partial Differential Equations, *Springer-Verlag*, Berlin and Heidelberg, 1968.

[22] H. Liu and H. Dai, Inverse problems for nonsymmetric matrices with a submatrix constraint, *Appl. Math. Comput.*, 189(2007), 1320-1330.

[23] V.L. Mehrmann, The Autonomous Linear Quadratic Control Problem, *Springer-Verlag*, Berlin and Heidelberg, 1991.

[24] I.R. Petersen, Disturbance attenuation and H^{∞}-optimization: A design method based on the algebraic Riccati equation, *IEEE Trans. Automat. Control*, 32(1987), 427-429.

[25] T. Rees, H.S. Dollar and A.J. Wathen, Optimal solvers for PDE-constrained optimization, *SIAM J. Sci. Comput.*, 32(2010), 271-298.

[26] Z.-R. Ren, On Preconditioned Iterative Methods for Sinc Systems of Linear Third-Order Ordinary Differential Equations, Ph.D. Thesis, *Academy of Mathematics and Systems Science, Chinese Academy of Sciences*, Beijing, P.R. China, 2011.

[27] A. van der Schaft, L_2-Gain and Passivity Techniques in Nonlinear Control, Second Edition, *Springer-Verlag*, London, 2000.

[28] B.F. Smith, P.E. Björstad and W.D. Gropp, Domain Decomposition: Parallel Multilevel Methods for Elliptic Partial Differential Equations, *Cambridge University Press*, Cambridge, 1996.

[29] A. Toselli and O.B. Widlund, Domain Decomposition Methods–Algorithms and Theory, *Springer-Verlag*, Berlin, 2005.

[30] M.-S. Wei, Perturbation theory for the Eckart-Young-Mirsky theorem and the constrained total least squares problem, *Linear Algebra Appl.*, 280(1998), 267-287.

[31] M.-S. Wei, Theory and Computations for Generalized Least-Squares Problems, *Science Press*, Beijing, 2006. (In Chinese)

[32] Y.-X. Yuan and H. Dai, Inverse problems for symmetric matrices with a submatrix constraint, *Appl. Numer. Math.*, 57(2007), 646-656.

[33] Y.-X. Yuan and H. Dai, The nearness problems for symmetric matrix with a submatrix constraint, *J. Comput. Appl. Math.*, 213(2008), 224-231.

[34] K.-M. Zhou and P.P. Khargonekar, An algebraic Riccati equation approach to H^∞ optimization, *Systems Control Lett.*, 11(1988), 85-91.